BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 5, September 1975

SPRINGER-TYPE THEOREMS FOR SPINOR GENERA OF QUADRATIC FORMS

BY A. G. EARNEST AND J. S. HSIA

Communicated by D. J. Lewis, May 5, 1975

At the Quadratic Forms Conference (Baton Rouge, Louisiana, 1972), N. C. Ankeny raised the question on the behaviour of the genus of a positive definite integral quadratic form upon inflation to a totally real number field. Here we announce some results of the closely related problem of how the *spinor* genus behaves when lifted to an overfield. We treat this question geometrically and also in a more general setting; namely, we study the spinor genera associated with an arbitrary quadratic lattice, not necessarily a free lattice. A classical theorem of Springer [**S**] asserts an anisotropic quadratic space over F remains anisotropic in E if the field degree $[E: F]$ is odd. In terms of classical Witt rings, it says the natural map $W(F) \rightarrow W(E)$ is injective. Our results for the spinor genus behaviour are similar in spirit. Detailed proofs will appear elsewhere. Unexplained notations are from [**O**].

Let E/F be a finite extension of global fields, $\mathcal{O}_E$, $\mathcal{O}_F$ the integers in E, F respectively, L a lattice on a regular space V over F with rank $r(L) \geqslant 3$. Put $\widetilde{V} = V \otimes E$ and $\widetilde{L} = L \otimes \mathcal{O}_E$. Define maps β: $J_F \rightarrow J_E$ and γ: $J_V \rightarrow J_{\widetilde{V}}$, respectively, by $(\beta(j_p))_P = j_p$ and $(\gamma(u_p))_P = u_p \otimes E_P$ for $P|p$. Then β and γ induce vertical maps ψ_L and Γ_L, respectively, in the commutative diagram

$$\begin{array}{ccc} J_V/P_V J'_V J_L & \rightarrowtail - - - - \twoheadrightarrow & J_F/P_D J_F^L \\ \Gamma_L \Big\downarrow & & \Big\downarrow \psi_L \\ J_{\widetilde{V}}/P_{\widetilde{V}} J'_{\widetilde{V}} J_{\widetilde{L}} & \rightarrowtail - - - - \twoheadrightarrow & J_E/P_{\widetilde{D}} J_E^{\widetilde{L}} \end{array}$$

so that Γ_L is injective if and only if ψ_L is injective as the horizontal maps are isomorphisms. Our main results are:

THEOREM A. *Let L be a quadratic lattice of rank $r(L) \geqslant 3$ and defined over a global field F with the property that at each dyadic localization L_p is modular. Then for any odd degree field extension E/F, ψ_L is injective.*

THEOREM B. *Let L be a quadratic lattice of rank $r(L) \geqslant 3$. If E/F is an odd degree field extension of number fields such that 2 is unramified in E, then ψ_L is injective.*

AMS (MOS) subject classifications (1970). Primary 10C05, 15A63; Secondary 10E45, 10B05, 20G30.

Key words and phrases. Integral quadratic forms, spinor genus, quadratic lattices, spinor norms.

REMARKS. Two important special cases where Theorem A directly applies are: (1) F is a function field and so has no dyadic spots, and (2) $F = \mathbf{Q}$, L an integral (with respect to scale) lattice with odd discriminant. When L is indefinite, the well-known Eichler-Kneser theorem asserts the (proper) spinor genus and the (proper) class coincide so that Theorems A and B then say nonequivalent proper classes in a given genus remain nonequivalent when lifted to E. A Hasse domain is a dedekind domain with quotient field a global field F and which can be obtained as the intersection of almost all valuation rings on F. Theorems A and B generalize to Hasse domains as well.

The proofs rely on rather long and meticulous studies of the spinor norms of integral rotations on the localizations L_p at the nonarchimedean spots, and how these groups behave going-up and coming down. The easiest cases are when p is nondyadic, where we employ a theorem of Kneser [K] who had calculated these groups at such spots. Dyadic theory is far more difficult. The conditions on Theorems A and B are such that heavy dependence is placed on the *exact* knowledge of $\theta(O^+(L_p))$ when: (i) p arbitrary dyadic but L_p modular; (ii) L_p arbitrary but p unramified dyadic. Computations for (ii) are more technically involved and we derive several Kneser-type theorems (see [K]). Aside from these rather intricate spinor considerations, the norm principles of Scharlau and Knebusch are needed as well as some lifting formulas for the Hilbert and Hasse symbols.

We now give examples which show the oddness of field degree is necessary in both Theorems A and B.

EXAMPLES. (1) $F = \mathbf{Q}(\sqrt{-5})$, $E = \mathbf{Q}(\sqrt{-5}, \sqrt{-1})$ its Hilbert class field, $L \cong \langle 1, 1, 1\rangle$. One can show the number $g^+(L)$ of proper spinor genera in the genus of L equals the order of the factor group C_F/C_F^2 where C_F is the ideal class group of F. Similarly, $g^+(\widetilde{L}) = |C_E/C_E^2|$. But, C_E is trivial. Hence, ψ_L must collapse. (2) $F = \mathbf{Q}$, $E = \mathbf{Q}(\sqrt{5})$, $L \cong \langle -1, 5^2 13^2, 5^4 13^4\rangle$. Then, $\theta(O^+(L_p)) = Q_2^*$ for $p = 2$; $= Q_5^{*2}, Q_{13}^{*2}$ for $p = 5$, 13 respectively; and $= U_p Q_p^{*2}$ for all other p's. Using these one shows $g^+(L) = 2$. While $g^+(\widetilde{L}) = 2$ also, ψ_L is *not* injective. Note that we may use the definite lattice $K \cong \langle 1, 5^2 13^2, 5^4 13^4\rangle$ and also conclude that ψ_K is not injective.

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DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43201