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GENERIC MATRICES, K_2, AND UNIRATIONAL FIELDS

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This note is to announce new examples of unirational fields, which are not rational, in all characteristics. Here a field K is called rational over k if it is a pure transcendental extension of k (with finitely many variables). It is called unirational if it is contained in a rational extension. The interesting case is when k is algebraically closed and in this case it is usually called the Luroth problem: Are unirational extensions rational?

Recently the Luroth problem was solved, almost simultaneously, by several people and we refer the reader to Deligne's article [3] surveying all these solutions. They are all examples of unirational irrational threefolds over C. However, Murre [5] extended Mumford's solution to all characteristics $\neq 2$, thereby showing that a nonsingular cubic hypersurface in $P^4(k)$ is irrational, where k is an algebraically closed field of characteristic $\neq 2$. Its unirationality is well known.

Our examples are of a completely different, totally algebraic, nature. They are based on Amitsur's proof that the generic division algebra is, in general, not a crossed product and on a recent theorem of S. Bloch [2] concerning the cokernel of the nth norm residue symbol from K_2/nK_2 to Br_n.

We now proceed to give some indication of the concepts and methods involved.

If k is a field we show first how to construct the "generic division algebra" over k. Let $x^i_{\alpha\beta}$ be n^2r independent variables, where $1 \leqslant \alpha$, $\beta \leqslant n$, $1 \leqslant i \leqslant r$, and $r > 1$. Let $X_i = (x^i_{\alpha\beta})$. These are called independent $n \times n$ generic matrices. The subring of $M_n(k(x^i_{\alpha\beta}))$ generated by these matrices, over k, is called the ring of generic matrices over k. It can be shown that every nonzero element of the ring of generic matrices is invertible in $M_n(k(x^i_{\alpha\beta}))$, and that the set of all fractions $g(X)^{-1} \cdot f(X)$ is a division subring of $M_n(k(x^i_{\alpha\beta}))$. This division ring has dimension n^2 over its center and therefore one is justified in naming it the generic division algebra.

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It is a nontrivial result of Amitsur [1] that the generic division algebra is not a crossed product for general n. By this we mean that it has no normal maximal subfield.

Next we discuss the norm residue symbol. This is explicitly described in Milnor's book [4], and is a map $R_{n,F}: K_2(F)/nK_2(F) \longrightarrow Br_n(F)$ for fields F containing n distinct roots of unity of order n. Taking F to be the center of the generic division algebra over, say, an algebraically closed field k of characteristic prime to n we see—using Amitsur's theorem—that $R_{n,F}$ is not surjective.

We now come to Bloch's theorem [2]. This states that the kernel and cokernel of $R_{n,k}$ and of $R_{n,k(t)}$ are, respectively, isomorphic. Starting with k algebraically closed we see that $\operatorname{coker}(R_{n,k}) = 0$. Thus $\operatorname{coker}(R_{n,k(t_1,\dots,t_m)}) = 0$ for every $m \geqslant 0$; taking F as above we see that it cannot be rational over k as $R_{n,F}$ has a nonzero cokernel. Finally it remains to show that F is unirational. This amounts to showing that F is a subfield of the center, $k(x^i_{\alpha\beta})$, of $M_n(k(x^i_{\alpha\beta}))$, and is easy.

ONE LAST REMARK. The general linear group $\mathrm{GL}_n(k)$ acts on $k(x^i_{\alpha\beta})$ with fixed field F. This makes our examples "homogeneous", an aspect not clear in the geometrical examples mentioned above.

A more detailed exposition of these results, including a complete presentation of the generic division algebra which is "polynomial identity free" will appear elsewhere.

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