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MULTIPLIERS OF CLOSED IDEALS OF $L^p(D^\infty)$

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Let G be a compact abelian group with character group X. Let E be a subset of X and for $1 \leqslant p \leqslant \infty$, let L_E^p be the ideal of E-spectral functions in $L^p(G)$. Let M_E^p be the space of complex-valued functions on E which multiply $\widehat{L_E^p}$ into itself, and let $M^p|_E$ be the set of restrictions to E of functions in M_X^p. We are interested in the following questions:

(i) Does $M_E^p = M^p|_E$?

(ii) Does an analogue of the Riesz-Thorin interpolation theorem hold for the spaces L_E^p? I.e., for $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, are the interpolation spaces obtained by applying Calderón's complex method of interpolation to $L_E^{p_1}$ and $L_E^{p_2}$ actually the intermediate L_E^p spaces?

(iii) For $1 \leqslant p < q < 2$ or $2 < q < p \leqslant \infty$, is $M_E^p \subseteq M_E^q$?

Question (i) is posed for the circle group $\mathbf{T}$ in [3, pp. 280-281] and has an affirmative answer for any G if $p = 2$ (trivially) or if $p = \infty$ (see [2, Theorem 3.3], [8], and, for a more general result, [6]). Question (ii) is inspired by [1, p. 344, Remarque], while (iii) seems natural in view of (i) and (ii). (An affirmative answer for either (i) or (ii) clearly implies the same for (iii).)

We take for our group G the Cantor group D^∞ $(= Z(2)^{\mathbf{N}})$ and we prove the following:

THEOREM 1. *There exists a subset E of X such that*:

(a) *for each $p \in [1, 2)$, there is a multiplier of $\widehat{L_E^p}$ into $\widehat{L_E^2}$ which is not in $M^p|_E$*;

(b) *the interpolation spaces B_t obtained by applying the complex method of interpolation to L_E^1 and L_E^2 are not the spaces L_E^p $(p = p(t) = 2/(2-t)$, $0 < t < 1)$.*

THEOREM 2. *For $p = 4, 6, \ldots$, there exists a subset $E_p \subseteq X$ such that $M_{E_p}^p \supsetneqq M^p|_{E_p}$.*

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If we consider the groups $Z(n)^{\mathbf{N}}$ for $n > 2$, we have the following more striking result.

THEOREM 3. *Fix* $n \in \{3, 4, \ldots\}$ *and let* $G = Z(n)^{\mathbf{N}}$. *There exists a subset* E *of* X *and a function* $m \in M_E^{2(n-1)}$ *which is in* M_E^p *for only a finite number of* $p \in (2, 2(n-1))$. *Thus* m *is in no space* $M^p|_E$ *for* $p \neq 2$.

REMARKS. (1) Our main tool is the following lemma, essentially due to Bonami [1, Chapitre III, Théorème 2, Lemma 1].

LEMMA 4. *For* $i = 1, 2$, *let* G_i *be a compact abelian group with character group* X_i, *let* $E_i \subseteq X_i$, *and let* m_i *be a multiplier of* $\widehat{L_{E_i}^{p_1}}$ *into* $\widehat{L_{E_i}^{p_2}}$, $1 \leq p_1 \leq p_2 < \infty$. *Then* $m_1 \cdot m_2$, *considered as a function on* $E_1 \times E_2 \subseteq X_1 \times X_2 = \widehat{G_1 \times G_2}$, *is a multiplier of* $\widehat{L_{E_1 \times E_2}^{p_1}}$ *into* $\widehat{L_{E_1 \times E_2}^{p_2}}$. *Further,* $\|m_1 \cdot m_2\| = \|m_1\| \cdot \|m_2\|$.

Our technique is the same for each of Theorems 1–3. We first obtain an example of slightly aberrant behavior in a finite group, and then we apply Lemma 4, in one way or another, to obtain the desired result. For the proof of Theorem 1, we also employ a result of Paley [7] in the manner of [4, Theorem B]. Part (b) of Theorem 1 is based on an elegant computation due to David N. Bock.

(2) Theorem 1 is true when D^∞ is replaced by an infinite product of finite commutative groups of bounded order. The proof of this theorem is essentially measure-theoretic, while the proofs of Theorems 2 and 3 depend more heavily on group-theoretic considerations.

(3) Part (a) of Theorem 1 is true if $1 \leq p < 4/3$ for any G such that X contains a $\Lambda(2)$ set which is not $\Lambda(4 + \epsilon)$ for any $\epsilon > 0$. (An easy proof of this observation may be deduced from [5, Theorem 6].)

ADDED IN PROOF. Since this paper was submitted, the author has proved Theorem 2 for any $p \in [1, 2) \cup \{4, 6, \cdots\}$ and for any infinite abelian group X.

REFERENCES

1. Aline Bonami, *Étude des coefficients de Fourier des fonctions de* $L^p(G)$, Ann. Inst. Fourier (Grenoble) **20** (1970), fasc. 2, 335–402 (1971). MR **44** #727.

2. B. Brainerd and R. E. Edwards, *Linear operators which commute with translations.* II. *Applications of the representation theorems,* J. Austral. Math. Soc. **6** (1966), 328–350. MR **34** #6543.

3. R. E. Edwards, *Fourier series: A modern introduction.* Vol. II, Holt, Rinehart and Winston, New York, 1967. MR **36** #5588.

4. A. Figà-Talamanca and G. I. Gaudry, *Multipliers of L^p which vanish at infinity*, J. Functional Analysis 7 (1971), 475–486. MR 43 #2429.

5. A. Figà-Talamanca and D. Rider, *A theorem of Littlewood and lacunary series for compact groups*, Pacific J. Math. 16 (1966), 505–514. MR 34 #6444.

6. I. Glicksberg and I. Wik, *Multipliers of quotients of L_1*, Pacific J. Math. 38 (1971), 619–624. MR 46 #5935.

7. R. E. A. C. Paley, *A remarkable series of orthogonal functions*. I, Proc. London Math. Soc. 34 (1932), 241–279.

8. J. H. Wells, *Restrictions for Fourier-Stieltjes transforms*, Proc. Amer. Math. Soc. 15 (1964), 243–246. MR 29 #1499.

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