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THE LEFSCHETZ NUMBER AND FIBER PRESERVING MAPS

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1. **Introduction.** Let $p\colon E \longrightarrow B$ be a Hurewicz fibration whose base B and fiber F are homotopy equivalent to a finite complex, and let $f\colon E \longrightarrow E$ be a fiber preserving map (over the identity). Let $d\colon \Omega B \longrightarrow F$ denote the transgression map which arises from the Puppe sequence of the fibration. Our purpose is to relate the homomorphisms induced by the projection p and the transgression d with the Lefschetz number of $g\colon F \longrightarrow F$, the restriction of f to the fiber.

THEOREM 1. *There is an S-map $\tau_f\colon B^+ \longrightarrow E^+$ such that the composite*

$$H^*(B) \xrightarrow{p^*} H^*(E) \xrightarrow{\tau_f^*} H^*(B)$$

is multiplication by the Lefschetz number Λ_g of g.

THEOREM 2. *For $k > 0$, $\Lambda_g d^* = 0\colon\ H^k(F) \longrightarrow H^k(\Omega B)$.*

Here H denotes singular cohomology with arbitrary coefficients.

We call τ_f a *transfer* map. It is a generalization of the transfer for coverings [6], [7] and for fiber bundles [1], [2], [3], [5]. A. Dold [4] has also defined transfer for a large category of maps.

The existence of the transfer leads to various restrictions on the algebraic invariants attached to a fibration with base and fiber a finite complex. In particular we have

COROLLARY 1. *Let $f\colon E \longrightarrow E$ be a fiber preserving map. Then*

(a) $p^* \otimes 1\colon\ h^*(B) \otimes Z[\Lambda_g^{-1}] \longrightarrow h^*(E) \otimes Z[\Lambda_g^{-1}]$ *is a split monomorphism for any cohomology theory h.*

(b) $\Lambda_g \partial_\#\colon\ \pi_n(B) \longrightarrow \pi_{n-1}(F)$ *is trivial, $n < 2$ (connectivity of F).*

We will outline two constructions of the transfer each of interest in its own right. The first involves a reduction of the fibration case to the smooth

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fiber bundle case. The second is a direct construction for fibrations, involving Spanier-Whitehead duality. Details of the results announced here and further applications will appear elsewhere.

2. **Smooth fiber bundles.** Let $p\colon E \longrightarrow B$ be a smooth fiber bundle with base B and fiber F closed manifolds, and let $f\colon E \longrightarrow E$ be a fiber preserving map. To construct the transfer in this case, choose an embedding $E \longrightarrow B \times R^s$ homotopic to p. Its normal bundle β is inverse to the bundle α of tangents along the fiber. Let $c\colon B^+ \wedge S^s \longrightarrow E^\beta$ denote the Pontryagin-Thom map, $\Delta\colon E \longrightarrow E^2$ the diagonal inclusion into the fiber square and $\pi_1\colon E^2 \longrightarrow E$ projection onto the first factor. Since $\Delta(E)$ has normal bundle α we have $c'\colon (E^2)^{\pi_1^*(\beta)} \longrightarrow E^{\alpha\otimes\beta} = E^+ \wedge S^s$. Define τ_f to be

$$B^+ \wedge S^s \xrightarrow{c} E^\beta \xrightarrow{\widetilde{(1,f)}} (E^2)^{\pi_1^*(\beta)} \xrightarrow{c'} E^+ \wedge S^s,$$

where $(1,f)\colon E \longrightarrow E^2$ sends e to $(e, f(e))$. Then τ_f meets the requirements of Theorem 1.

3. **Fiber smoothing theorems.** Let $F \longrightarrow E \longrightarrow B$ be a fibration such that B is a closed smooth manifold and F is a finite complex. We have

THEOREM 3 (*Open fiber smoothing theorem*). *There exists an open regular neighborhood U of F and a smooth fiber bundle $U \longrightarrow E' \longrightarrow B$ which is fiber homotopy equivalent to $F \longrightarrow E \longrightarrow B$.*

Let T^n denote the n-dimensional torus, $n = \dim(B)$.

THEOREM 4 (*Closed fiber smoothing theorem*). *There exists a closed regular neighborhood W of F and a smooth fiber bundle $W \times T^n \longrightarrow E' \longrightarrow B$ which is fiber homotopy equivalent to $F \times T^n \longrightarrow E \times T^n \longrightarrow B$.*

The extension of the transfer from smooth fiber bundles to fibrations is accomplished by the closed fiber smoothing theorem, which asserts that any fibration $p\colon E \longrightarrow B$, as above, is a retract of a smooth fiber bundle $p'\colon E' \longrightarrow B$. Let $E \xrightarrow{\lambda} E' \xrightarrow{\rho} E$ denote the asserted inclusion and retraction. If $f\colon E \longrightarrow E$ is fiber preserving, τ_f is then defined to be $\rho\tau_{\lambda f\rho}$.

The general case, i.e. where B is a finite complex can be reduced to the above case by embedding B as a retract of a closed smooth manifold.

4. **Duality.** A direct construction of transfer for fibrations is based on the following observation. Let $\mu\colon S^s \longrightarrow \hat{F} \wedge F^+$ be a duality map. Then $\hat{F} \wedge F^+$ is self dual in a canonical way, so that μ has a $2s$-dual $\hat{\mu}\colon \hat{F} \wedge F^+ \longrightarrow S^s$.

LEMMA. *Let g: $F \longrightarrow F$. The composite* $S^s \xrightarrow{\mu} \hat{F} \wedge F^+ \xrightarrow{1 \wedge g^+} \hat{F} \wedge F^+ \xrightarrow{\hat{\mu}} S^s$ *has degree* Λ_g.

By an ex-fibration we mean an object $E = (E, B, p\Delta)$ where p: $E \longrightarrow B$ is a fibration, Δ is a cross section, and (a) p has a lifting function with the property that if σ is a path in B its lifting in E which begins at $\Delta(\sigma(0))$ is $\Delta\sigma$. (b) $E \times \{0\} \cup \Delta(B) \times I$ is a vertical retract of $E \times I$. If p: $E \longrightarrow B$ is a fibration we have an ex-fibration $\bar{E}$. The disjoint union of E and B, with p: $E \longrightarrow B$ and $\bar{\Delta}$: $B \longrightarrow \bar{E}$ the obvious maps.

THEOREM 5. *If E is an ex-fibration there exists for some integer s, an ex-fibration E and an ex-map μ: $S^s \times B \longrightarrow \hat{E} \wedge_B \bar{E}$ such that the restriction of μ to each fiber is a duality map.*

Once the existence of dual ex-fibrations is established the remaining aspects of Spanier-Whitehead duality carry over easily to ex-fibrations over a fixed space B. To define transfer for a fibration p: $E \longrightarrow B$ and fiber preserving map f: $E \longrightarrow E$ let $\hat{E}$ be dual to $\bar{E}$. We have

$$S^s \times B \xrightarrow{\mu} \hat{E} \wedge_B \bar{E} \xrightarrow{1 \wedge (f, 1)} \hat{E} \wedge_B \bar{E} \xrightarrow{\hat{\mu} \wedge 1} S^s \wedge_B \bar{E}.$$

Identifying the section to a point on each side yields τ_f: $S^s \wedge B^+ \longrightarrow S^s \wedge E^+$. It follows from the Lemma that τ_f meets the requirements of Theorem 1.

REFERENCES

1. J. C. Becker, *Characteristic classes and K-theory*, Proceedings of the Binghampton New York Conference on Topology (to appear).
2. J. C. Becker and D. H. Gottlieb, *The transfer map and fiber bundles*, Topology 211 (1974), 277–288.
3. ———, *Applications of the evaluation map and transfer map theorems*, Math. Ann. (to appear).
4. A. Dold, *Transfert des points fixes d'une famille continue d'applications*, C. R. Acad. Sci. Paris (to appear).
5. D. H. Gottlieb, *Fiber bundles and the Euler characteristic*, J. Differential Topology (to appear).
6. D. S. Kahn and S. B. Priddy, *Applications of the transfer to stable homotopy theory*, Bull. Amer. Math. Soc. 78 (1972), 981–987. MR 46 #8220.
7. F. W. Roush, *Transfer in generalized cohomology theories*, Thesis, Princeton University, Princeton, N. J., 1971.

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