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# ROUND HANDLES AND HOMOTOPY OF NONSINGULAR VECTOR FIELDS

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**Introduction.** We consider nonsingular vector fields on compact connected $C^\infty$ manifolds. The question is: What orbit structures occur in which homotopy classes of nonsingular vector fields? We show that in dimensions 4 and greater, a nonsingular Morse-Smale (NMS) vector field occurs in each homotopy class. Under the additional assumption that the first Betti number of the manifold is nonzero, we show that a nonsingular volume-preserving (NVP) vector field occurs in each homotopy class. These results are based on the round handle decomposition theorem, interesting in its own right as a structure theorem for manifolds whose Euler characteristic is 0.

Let $M$ be a compact manifold whose boundary has been partitioned into two unions of components: $\partial M = \partial_- M \cup \partial_+ M$, $\partial_- M \cap \partial_+ M = \emptyset$. Then the following are equivalent:

1. $\chi(\partial_- M) = \chi(M)$.
2. $\chi(\partial_+ M) = \chi(M)$.
3. There exists a nonsingular vector field on $M$ pointing inward on $\partial_- M$ and outward on $\partial_+ M$.

DEFINITION. The pair $(M, \partial_- M)$ will be called a *flow manifold* if 1, 2 and 3 above are true. This does not exclude the possibility, of course, that $\partial_- M$, $\partial_+ M$, or $\partial M$ may be empty.

DEFINITION. A nonsingular Morse-Smale (NMS) vector field $V$ on the flow manifold $(M, \partial_- M)$ is one which satisfies (a), (b) and (c) below:

(a) $V$ has nonwandering set equal to a finite number of closed orbits, each having a hyperbolic Poincaré map.

(b) The stable manifold (inset) of one closed orbit is transversal to the unstable manifold (outset) of any other closed orbit.

(c) $V$ points inward on $\partial_- M$ and outward on $\partial_+ M$.

DEFINITIONS. A *round handle of index $k$* (and dimension $n$) is a copy

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of $R_k = S^1 \times D^k \times D^{n-k-1}$. The *attaching region* $\partial_- R_k$ of $R_k$ is the submanifold $S^1 \times S^{k-1} \times D^{n-k-1}$ of $\partial R_k$, where $S^{k-1} = \partial D^k$.

THEOREM 1 (ROUND HANDLE DECOMPOSITION THEOREM). *Let $(M^n, \partial_- M)$ be a flow manifold with $n \geqslant 4$. Then $M$ admits a decomposition of the form*

$$M = (\partial_- M \times I) + R_0^1 + \cdots + R_0^{\alpha_0} + R_1^1 + \cdots + R_{n-1}^{\alpha_{n-1}}.$$

This means that each $R_k^i$ is attached via a diffeomorphism of $\partial_- R_k^i$ to the boundary of the previous stuff, but never to $\partial_- M \times \{0\}$. If we further assume that $\partial_- M \neq \varnothing$ and $\partial_+ M \neq \varnothing$, we may write

$$M = (\partial_- M \times I) + R_1^1 + \cdots + R_{n-2}^{\beta_{n-2}},$$

i.e. there exists a round handle decomposition avoiding round handles of extreme indices.

COROLLARY 1. *Let $(M^n, \partial_- M)$ be a flow manifold with $n \geqslant 4$. Then $(M, \partial_- M)$ admits an NMS vector field. If we further assume that $\partial_- M \neq \varnothing$ and $\partial_+ M \neq \varnothing$, then $(M, \partial_- M)$ possesses an NMS vector field with no source or sink closed orbits.*

THEOREM 2. *Let $V$ be a nonsingular vector field on the flow manifold $(M^n, \partial_- M)$ pointing inward on $\partial_- M$ and outward on $\partial_+ M$. Assume $M$ is orientable and $n \geqslant 4$. Then $V$ is homotopic* rel $\partial M$ *(through nonsingular vector fields) to an NMS vector field. If further $\partial_- M \neq \varnothing$ and $\partial_+ M \neq \varnothing$ then $V$ is homotopic* rel $\partial M$ *to an NMS vector field with no sources or sinks.*

THEOREM 3. *Let $M^n$ be compact, connected, and orientable with $\chi(M) = 0$, $n \geqslant 4$, and $H^1(M) \neq 0$. Then any nonsingular vector field $V$ on $M$ is homotopic to a field $V_1$ which preserves some smooth nonzero volume form $\omega$ on $M$ (which may be preassigned).*

PROOF (SKETCH). Using $H^1(M) \neq 0$ we first find a compact, connected, oriented submanifold $i$: $N^{n-1} \subseteq M$ with $i_*[N] \neq 0$ in $H_{n-1}(M)$. By surgering $N$ we may find a homologous submanifold $N'$ and a vector field $V'$ homotopic to $V$ such that $V'$ is nowhere tangent to $N'$. Then cutting $M$ open along $N'$ gives us a flow manifold $\widetilde{M}$, and $V'$ becomes a vector field $\widetilde{V}$ on $\widetilde{M}$. By Theorem 2, $\widetilde{V}$ is homotopic rel $\partial\widetilde{M}$ to an NMS vector field $\widetilde{V}_1$ on $\widetilde{M}$ having no sources or sinks. This last property permits finding (one round handle at a time) a smooth nonzero volume $\widetilde{\omega}$ on $\widetilde{M}$ preserved by $\widetilde{V}_1$. By carefully regluing

$\widetilde{M}$ to obtain $M$ once again, $\widetilde{V}_1$ becomes a smooth nonsingular vector field $V_1$ (which is not NMS), and $\widetilde{\omega}$ becomes a smooth nonzero volume $\omega$ on $M$ preserved by $V_1$. The form $\omega$ may as well have been preassigned, since Moser [4] shows any two volume forms are equivalent up to constant multiple under a diffeomorphism of $M$ isotopic to the identity.

**Remark.** Theorem 3 fails on the 2-torus $T^2$.

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