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THE NUMBER OF ZEROES OF AN ANALYTIC FUNCTION IN A CONE

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It is not possible to estimate the number of zeroes of an analytic function of several variables defined in a cone by reducing the problem to the 1-dimensional case via Crofton's formula or similar tools of Nevanlinna theory (see e.g. [4]). We propose to extend the classical result due to Pfluger and Levin [3] using a potential theory approach.

Let S^{m-1} be the unit sphere in the euclidean space $\mathbf{R}^m$, D an open subset of S^{m-1}, ∂D smooth and of bounded curvature. For $0 < r \leqslant \infty$ we set $D_r = \{tx: x \in D, 0 < t < r\}$. Denote by $\rho_1 = \rho_1(D)$ the positive number such that $\rho_1(\rho_1 + m - 2)$ is the first eigenvalue of the Laplace-Beltrami operator in D for the Dirichlet problem. Thus we have

THEOREM. *Let u be a subharmonic function in D_∞, such that $u \not\equiv -\infty$, $u(x) \leqslant A + B|x|^\rho$ for every $x \in D_\infty$. If $\rho > \rho_1$, D' is an arbitrary open set, $\bar{D}' \subseteq D$, then there exists a constant $C = C(D', \rho)$ such that*

$$\varlimsup_{r \to \infty} r^{-\rho - m + 2} \int_{D'_r} \Delta u \leqslant CB.$$

If we identify C^n with $\mathbf{R}^{2n}$ and f is an analytic function in D, then $\log|f(z)|^2$ is subharmonic and

$$\sigma_D(r) = \frac{(n-1)!}{2\pi^n} \int_{D_r} \Delta \log|f(z)|^2$$

represents the euclidean area of the variety $\{z \in D_r: f(z) = 0\}$. For $n = 1$, it is just the number of zeroes of f in D_r; see [2]. Therefore we obtain the following

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COROLLARY. *Let f be a nonzero analytic function in D_∞ satisfying $|f(z)| \leqslant A \exp(B|z|^\rho)$ for some $\rho > \rho_1$; then for any D' open, $\bar{D}' \subseteq D$, we have*

$$\overline{\lim_{r\to\infty}}\, \sigma_{D'}(r) r^{-\rho-m+2} \leqslant CB.$$

The details of the proof and related results appear in [1], here we just present the bare bones of the proof of the theorem. First, we can show that

(i) the harmonic measure of $S_r = \{rx: x \in D\}$ at a fixed point x_0, behaves like $O(r^{-\rho_1})$,

(ii) if $G_r(x)$ is the Green function of D_r with pole at x_0, $0 < \epsilon \leqslant 1$ fixed, then

$$G_r(x) \geqslant \text{const}\, r^{-\rho_1 - m + 2}, \qquad r \longrightarrow \infty$$

for $x \in D'_{\epsilon r}$, $|x| > r_0$,

(iii) we can reduce the general case to the one in which $u \leqslant 0$ on ∂D_∞.

Then we apply Green's formula, assuming $u(x_0) \neq -\infty$,

$$\int_{D_r} G_r(x)\Delta u = -u(x_0) + \int_{\partial D_r} u(x)\frac{\partial G_r}{\partial \nu}(x)$$

($\partial/\partial\nu$ derivative in the direction of the inner normal). By (i), (iii) and the assumption on u we have

$$\int_{D_r} G_r(x)\Delta u = O(r^{\rho-\rho_1}) \quad \text{as } r \longrightarrow \infty.$$

Applying (ii), the conclusion of the Theorem follows.

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