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# COMPACT ANR'S HAVE FINITE TYPE[1]

BY JAMES E. WEST

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In 1950, J. H. C. Whitehead [15] proved that compact metric ANR's have the homotopy types of (countable) cell complexes and asked whether they have the homotopy types of finite-dimensional cell complexes. In 1954 in his talk at the International Congress of Mathematicians in Amsterdam [2], Borsuk asked whether they have the homotopy types of finite complexes. The simply-connected case was answered in 1957 by de Lyra [9], and in 1965, Wall [14] produced an applicable obstruction theory for finiteness. That year, Mather settled the case of products of $S^1$ [10], and a year later, Gersten's product formula [5] for Wall's obstruction settled the case of products with polyhedra having zero Euler characteristic. Then Siebenmann proved it for a large class of (finite-dimensional) manifolds [13] and with Kirby extended this to all (finite-dimensional, compact) manifolds [12]. Finally, in 1973 Chapman [3] added Hilbert cube manifolds and locally triangulable spaces.

In this notice an affirmative solution of these questions is sketched, using the notions of cell-like mappings and Hilbert cube manifolds together with a recent result of R. Miller [11], which is a cell-like analog of Mather's theorem and provides the basic existence theorem for cell-like mappings used here.

A cell-like ($CE$) mapping is a proper surjection such that each point-inverse has the shape of a point. Cell-like mappings between locally compact, separable metric ANR's are homotopy equivalences [6], [7], [8]. The strategy of this argument is to construct, for any compact ANR $A$ a $CE$ mapping $f$: $M^Q \longrightarrow A$ from some Hilbert cube manifold $M^Q$. Because $M^Q$ is homeomorphic, by Chapman's Triangulation Theorem, [3], to $K \times Q$ for some finite complex $K$ ($Q$ is the Hilbert cube), $A$ must have the homotopy type of

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$K$. To obtain this $CE$ map from Miller's result that (essentially) $A \times S^1$ and the cone $C(A)$ of $A$ are $CE$-images of Hilbert cube manifolds (finite complexes in the finite-dimensional case), the global property is reduced to a local one about Hilbert cube manifold factors which allows the restriction of attention to the base of the cone $C(A)$. The first step is a mapping cylinder theorem.

THEOREM 1. *Let* $f: M^Q \longrightarrow A$ *be a* $CE$ *mapping of a Hilbert cube manifold to a compact* ANR $A$. *Then there is a homeomorphism from* $M^Q$ *to the mapping cylinder* $M_f$ *of* $f$ *which is homotopic to the natural inclusion* $i: M^Q \longrightarrow M_f$ *of* $M^Q$ *into* $M_f$.

This provides a link between $CE$ mappings and Hilbert cube manifolds. A subset $X$ of an ANR $Y$ is said to have Property Z in $Y$ if it is closed and if for all open sets $U$ of $Y$ the inclusion $U \backslash X \longrightarrow U$ is a homotopy equivalence. If $Y$ is a Hilbert cube manifold this is equivalent to the property that $X$ lies in a collared submanifold. In the mapping cylinder $M_f$ of Theorem 1, the base (the natural copy of $A$) has Property Z, a result obtained from an observation [4] of Doug Curtis—that the base $A$ has Property Z in $M_f$ if and only if $f$ is a "fine homotopy equivalence", i.e. for every open cover $\mathcal{U}$ of $A$, there is a homotopy inverse $g$ of $f$ such that the homotopies to the identity of $fg$ and $gf$ may be limited by $\mathcal{U}$ and $f^{-1}(\mathcal{U})$—and the result of Haver [6] that $f$ is a fine homotopy equivalence if and only if $CE$ (see also Lacher [7], [8]). With a little Hilbert cube manifold theory, this and Theorem 1 yield the local characterization of ANR $CE$ images of compact Hilbert cube manifolds.

THEOREM 2. *For a compact metric* ANR $A$, *the following are equivalent*:

(1) $A$ *is the* $CE$ *image of a Hilbert cube manifold*;

(2) *the union of any two Hilbert cube manifolds along a copy of* $A$ *which has Property* Z *in each is a Hilbert cube manifold factor*; *and*

(3) *there exist two Hilbert cube manifolds whose union along a common copy of* $A$ *is a Hilbert cube manifold factor.*

(The space $X$ is here defined to be a Hilbert cube manifold factor if $X \times Q$ is a Hilbert cube manifold.) Coupling Theorem 2 with Miller's theorem and regarding $C(A)$ as $A \times [0, 1]/A \times \{0\}$, it is easy to demonstrate, using the Homogeneity Theorem of Anderson and Chapman [1], that

THEOREM 3. *Every compact metric ANR is the CE image of a Hilbert cube manifold.*

This has, as indicated above, the following corollary:

COROLLARY. *Every compact metric ANR is homotopy-equivalent to a compact polyhedron.*

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DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14850