

A REMARK ON SIMPLY-CONNECTED 3-MANIFOLDS

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In this note we will describe an “infinite process”, related to “wild topology”, with applications to closed, smooth manifolds.

DEFINITION. Consider an increasing sequence of solid tori:

$$T_1 \hookrightarrow T_2 \hookrightarrow \cdots \hookrightarrow T_n \hookrightarrow T_{n+1} \hookrightarrow \cdots$$

($T_i = k(i) \# (S^1 \times D^2)$) such that:

- (a) $T_k \subset \text{int } T_{k+1}$, and T_k is a smooth submanifold of T_{k+1} .
- (b) The natural inclusion $T_k \xrightarrow{j_k} T_{k+1}$ is null-homotopic.

The open 3-manifold $W = \varinjlim T_i$ will be called a Whitehead manifold.

It is an easy (and well-known) exercise to show that for any Whitehead manifold W , one has: $W \times R = R^4$.

THEOREM 1. *Let X be a smooth 3-manifold with $\pi_1 X = 0$, T a solid torus, and $T \xrightarrow{j} X$ a smooth embedding. There exists a Whitehead manifold W defined by a sequence of nested tori:*

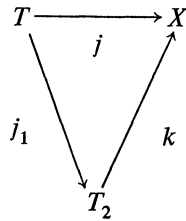
$$T = T_1 \xrightarrow{j_1} T_2 \xrightarrow{j_2} \cdots$$

and a smooth embedding $W \xrightarrow{k} X$ such that the following diagram is commutative:

$$\begin{array}{ccc}
 T & \xrightarrow{j} & X \\
 \downarrow & & \uparrow k \\
 \cdots \circ j_2 \circ j_1 & & W
 \end{array}$$

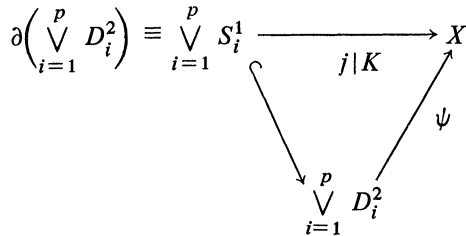
PROOF. It suffices to show that there exists a solid torus T_2 and a commutative diagram of smooth embeddings:

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such that j_1 is null-homotopic. (Afterwards we can continue this process indefinitely.)

Consider a wedge (bouquet) of circles $K = \bigvee_{i=1}^p S_i^1$ which is a spine of T . There exists a commutative diagram:



where ψ is a generic immersion, without triple points, such that the set of double points is a union of (disjoined) arcs.

Take $T_2 =$ a regular neighborhood of $\psi(\bigvee_{i=1}^p D_i^2)$ in X (containing T in its interior) a.s.o.

COROLLARY 2. *Let Σ^3 be a smooth homotopy 3-sphere. There exist two open subsets $U_1, U_2 \subset \Sigma^3 \times R$ such that:*

- (a) $\Sigma^3 \times R = U_1 \cup U_2,$
- (b) U_i is diffeomorphic to R^4 . \square

PROOF. Let $\Sigma^3 = T' \cup T''$ be a Heegaard decomposition, and consider the Whitehead manifolds W', W'' (containing T', T'' and contained in Σ^3) provided by Theorem 1.

Since $W \times R = R^4$ for any Whitehead manifold, we can take $U_1 = W' \times R \subset \Sigma^3 \times R, U_2 = W'' \times R \subset \Sigma^3 \times R$. Q.E.D.