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A REMARK ON SIMPLY-CONNECTED 3-MANIFOLDS

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In this note we will describe an "infinite process", related to "wild topology", with applications to closed, smooth manifolds.

DEFINITION. Consider an increasing sequence of solid tori:

$$T_1 \hookrightarrow T_2 \hookrightarrow \cdots \hookrightarrow T_n \hookrightarrow T_{n+1} \hookrightarrow \cdots$$

($T_i = k(i) \# (S^1 \times D^2)$) such that:

(a) $T_k \subset \operatorname{int} T_{k+1}$, and T_k is a smooth submanifold of T_{k+1}.

(b) The natural inclusion $T_k \longrightarrow_{j_k} T_{k+1}$ is null-homotopic.

The *open* 3-manifold $W = \varinjlim T_i$ will be called a Whitehead manifold. It is an easy (and well-known) exercise to show that for any Whitehead manifold W, one has: $W \times R = R^4$.

THEOREM 1. *Let X be a smooth 3-manifold with $\pi_1 X = 0$, T a solid torus, and $T \longrightarrow_j X$ a smooth embedding. There exists a Whitehead manifold W defined by a sequence of nested tori*:

$$T = T_1 \xhookrightarrow{j_1} T_2 \xhookrightarrow{j_2} \cdots$$

and a smooth embedding $W \longrightarrow_k X$ such that the following diagram is commutative:

$$\begin{array}{ccc} T & \xrightarrow{\quad j \quad} & X \\ & {\scriptstyle \cdots \circ j_2 \circ j_1} \searrow \quad \nearrow {\scriptstyle k} & \\ & W & \end{array}$$

PROOF. It suffices to show that there exists a solid torus T_2 and a commutative diagram of smooth embeddings:

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$$\begin{array}{ccc} T & \xrightarrow{\quad j \quad} & X \\ & {\scriptstyle j_1} \searrow \quad \nearrow {\scriptstyle k} & \\ & T_2 & \end{array}$$

such that j_1 is null-homotopic. (Afterwards we can continue this process indefinitely.)

Consider a wedge (bouquet) of circles $K = \bigvee_{i=1}^{p} S_i^1$ which is a spine of T. There exists a commutative diagram:

$$\begin{array}{ccc} \partial\left(\bigvee_{i=1}^{p} D_i^2\right) \equiv \bigvee_{i=1}^{p} S_i^1 & \xrightarrow{\quad j \mid K \quad} & X \\ & \searrow \quad \nearrow {\scriptstyle \psi} & \\ & \bigvee_{i=1}^{p} D_i^2 & \end{array}$$

where ψ is a generic immersion, without triple points, such that the set of double points is a union of (disjoined) arcs.

Take T_2 = a regular neighborhood of $\psi(\bigvee_{i=1}^{p} D_i^2)$ in X (containing T in its interior) a.s.o.

COROLLARY 2. *Let Σ^3 be a smooth homotopy 3-sphere. There exist two open subsets $U_1, U_2 \subset \Sigma^3 \times R$ such that*:

(a) $\Sigma^3 \times R = U_1 \cup U_2$,

(b) U_i *is diffeomorphic to* R^4. □

PROOF. Let $\Sigma^3 = T' \cup T''$ be a Heegaard decomposition, and consider the Whitehead manifolds W', W'' (containing T', T'' and contained in Σ^3) provided by Theorem 1.

Since $W \times R = R^4$ for any Whitehead manifold, we can take $U_1 = W' \times R \subset \Sigma^3 \times R$, $U_2 = W'' \times R \subset \Sigma^3 \times R$. Q.E.D.

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