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CONTINUOUS DIFFERENTIABILITY OF THE FREE BOUNDARY FOR WEAK SOLUTIONS OF THE STEFAN PROBLEM

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ABSTRACT. We announce a result concerning the continuous differentiability of the unknown boundary curve defined by a weak solution of the one-dimensional two-phase Stefan problem.

We deal with the following two-phase Stefan problem: to determine $u(x, t)$ for $0 \leqq t \leqq T$, $0 \leqq x \leqq 1$ and $s(t)$ for $0 \leqq t \leqq T$ such that (i) $0 < s(t) < 1$, $s(0)=b$; (ii) $u_t=\beta_1 u_{xx}$ for $0<t\leqq T$, $0<x<s(t)$ and $u_t=\beta_2 u_{xx}$ for $0<t\leqq T$, $s(t)<x<1$; (iii) $u(0,t)=f_1(t)>0$ and $u(1,t)=f_2(t)<0$ for $0\leqq t\leqq T$; (iv) $u(x,0)=\psi(x)$ for $0\leqq x\leqq 1$; (v) $u(s(t),t)=0$ for $0\leqq t\leqq T$; and (vi) $\alpha\dot{s}(t)=-u_x(s(t)-0,t)+u_x(s(t)+0,t)$ for $0<t\leqq T$. Here and throughout, β_i and α are positive parameters, $b\in(0,1)$, f_i and ψ are continuous functions with $f_1(0)=\psi(0)$, $f_2(0)=\psi(1)$, $\psi(b)=0$, $\psi(x)>0$ for $0\leqq x<b$, $\psi(x)<0$ for $b<x\leqq 1$, and $|\psi(x)|\leqq K|b-x|$ for $0\leqq x\leqq 1$.

Cannon and Primicerio [3], following the work of Cannon, Douglas and Hill [2] showed that this problem has a unique *classical solution* (one for which the expressions appearing in (vi) are defined and continuous for $0<t\leqq T$) on condition that the f_i and ψ are bounded by certain constants which depend on the parameters of the problem. A. Friedman [4],

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following the work of S. L. Kamenomostskaja [6], introduced the notion of a *weak solution* of a Stefan problem and established existence and uniqueness theorems (in n-dimensions) for such solutions. In [5], he considered the case $n=1$ in more detail and obtained certain continuity properties of weak solutions. However, the question of whether weak solutions are actually classical remained open. The objective of the present work is to provide an affirmative answer to this question.

Friedman's definition of weak solution, specialized to the problem under consideration, is as follows:

DEFINITION. *A bounded measurable function $u(x, t)$ in $\Omega_T=\{(x, t);\ 0<t\leqq T,\ 0<x<1\}$ is called a* weak solution *if, for all $\phi \in C^{2,1}(\bar{\Omega}_T)$ such that $\phi=0$ for $x=0, 1$ and $t=T$, we have*

$$\int_0^T\int_0^1 [u\phi_{xx} + a(u(x, t))\phi_t]\, dx\, dt$$
$$= -\int_0^1 a(\psi(x))\phi(x, 0)\, dx + \int_0^T [f_2(t)\phi_x(1, t) - f_1(t)\phi_x(0, t)]\, dt$$

where

$$\begin{aligned} a(u(x, t)) &= \beta_1 u(x, t) && \text{where } u(x, t) > 0, \\ &= -\alpha + \beta_2 u(x, t) && \text{where } u(x, t) < 0, \\ &= \text{some measurable function of } (x, t) \text{ with} \\ &\qquad\qquad \text{range } \subseteq [-\alpha, 0] && \text{where } u(x, t) = 0. \end{aligned}$$

The results of Friedman are summarized in the following

THEOREM 1 (A. FRIEDMAN [5]). *Assume that $\psi' \in L_2(0, 1)$ and that there exists $\Psi(x, t)$ in $\bar{\Omega}_T$ with $\Psi_x, \Psi_{xx}, \Psi_t$ continuous in $\bar{\Omega}_T$ and $\Psi(0, t)=f_1(t)$, $\Psi(1, t)=f_2(t)$, $\Psi(x, 0)=\psi(x)$ for $0\leqq t\leqq T$ and x close to 0 or 1. Then:*

(a) *There exists a unique weak solution $u(x, t)$.*

(b) *$u(x, t)$ is continuous on $\bar{\Omega}_T$, smooth in $\Omega_T\backslash\{u=0\}$, and satisfies $u_t=\beta_1 u_{xx}$ $[u_t=\beta_2 u_{xx}]$ in $\Omega_T\cap\{u>0\}$ [resp. $\Omega_T\cap\{u<0\}$].*

(c) *There exists $d>0$, which depends only on the data, such that $u>0$ $[u<0]$ for $|x|\leqq d$ [resp. $|1-x|\leqq d$].*

(d) *There exists a constant C such that*

$$\left[\int_0^1 |u_x(x, t)|^2\, dx\right]^{1/2} \leqq C \qquad (0 \leqq t \leqq T).$$

(e) *For each $t \in [0, T]$ and $x_1, x_2 \in [0, 1]$,*

$$|u(x_1, t) - u(x_2, t)| \leqq C\, |x_1 - x_2|^{1/2}.$$

(f) *For each $t \in [0, T]$, there exists a unique $s(t)$ such that $u(s(t), t)=0$.*

(g) *$s(t)$ is a continuous function of t for $0\leqq t<T$.*

(h) *The energy relation*

$$
\begin{aligned}
\alpha[s(t)-b] = \beta_1^{-1}\int_0^b \psi(x)\,dx + \beta_2^{-1}\int_b^1 \psi(x)\,dx - \beta_1^{-1}\int_0^{s(t)} u(x,t)\,dx \\
-\beta_2^{-1}\int_{s(t)}^1 u(x,t)\,dx + \int_0^t [u_x(1,\tau)-u_x(0,\tau)]\,d\tau
\end{aligned} \tag{1}
$$

is satisfied for each $t \in [0, T]$.

We may now state our main result.

THEOREM 2. *Let* $u(x, t)$ *and* $s(t)$ *satisfy* (b)–(h) *of Theorem* 1, *and assume* (*without loss of generality*) *that* $s(t)$ *is continuous for* $0 \leqq t \leqq T$. *Then* $s(t)$ *is continuously differentiable for* $0 < t \leqq T$, $u_x(s(t) \pm 0, t)$ *are well defined, bounded and continuous for* $0 < t \leqq T$ *and*

$$\alpha\dot{s}(t) = -u_x(s(t)-0,t) + u_x(s(t)+0,t) \qquad (0 < t \leqq T). \tag{2}$$

In particular, Theorems 1 and 2 guarantee the existence of a unique classical solution without the size restrictions on the data which were imposed in [**3**].

The first step in the proof of Theorem 2 is to show that $s(t)$ satisfies a Hölder condition with exponent 3/4.

LEMMA. *Let* $u(x, t)$ *and* $s(t)$ *be as in Theorem* 2. *Then there exists a constant* M *such that*

$$|s(t_1)-s(t_2)| \leqq M\,|t_1-t_2|^{3/4} \qquad (0 \leqq t_1, t_2 \leqq T). \tag{3}$$

The Lemma puts us in position to utilize the following result concerning the boundary behavior of the derivative of a solution of the heat equation which vanishes on a nonsmooth boundary curve.

THEOREM 3. *Let* $s(t)$ *be such that* $s(t) \geqq d > 0 \quad (0 \leqq t \leqq T)$, $s(0) = b$ *and*

$$|s(t_1)-s(t_2)| \leqq M\,|t_1-t_2|^{\lambda} \qquad (0 \leqq t_1, t_2 \leqq T)$$

where $\frac{1}{2} < \lambda \leqq 1$. *Let* $v(x, t)$ *be the solution of the problem* (i) $v_t = v_{xx}$ $(0 < x < s(t),\ 0 < t \leqq T)$; (ii) $v(x, 0) = \phi(x)$ $(0 \leqq x \leqq b)$; (iii) $v(0, t) = f(t)$ $(0 \leqq t \leqq T)$; (iv) $v(s(t), t) = 0$ $(0 \leqq t \leqq T)$; *where* $f(t)$ *and* $\phi(x)$ *are continuous with* $f(0) = \phi(0)$ *and* $|\phi(x)| \leqq K(b-x)$ $(0 \leqq x \leqq b)$. *Then* $v_x(x, t)$ *converges as* $x \to s(t)-$ *to a limit* $v(s(t)-0, t)$ *which is a bounded continuous function of* t *for* $0 < t \leqq T$. *Moreover, the convergence is uniform on* $[\delta, T]$ *for any* $\delta > 0$.

To complete the proof of Theorem 2, it is sufficient to show that

$$\alpha[s(t_2) - s(t_1)] = \int_{t_1}^{t_2} [-u_x(s(t) - 0, t) + u_x(s(t) + 0, t)]\, dt \tag{4}$$

for any $t_1, t_2 \in (0, T]$. This is done by using the energy relation together with the uniform convergence result of Theorem 3.

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