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DUALITY IN CROSSED PRODUCTS AND VON NEUMANN ALGEBRAS OF TYPE III

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In this paper, we announce further results succeeding to the previous papers [2] and [3]. Suppose $\mathcal{M}$ is a von Neumann algebra and G an abelian locally compact group. Let $\sigma : t \in G \to \sigma_t \in \mathrm{Aut}(\mathcal{M})$ be a continuous homomorphism in the sense that for each $x \in \mathcal{M}$, the map: $t \in \boldsymbol{R} \to \sigma_t(x) \in \mathcal{M}$ is σ-weakly continuous. We construct the crossed product $\mathcal{M} \otimes_\sigma G$ of $\mathcal{M}$ by G with respect to σ. In [2], we have shown that there is a canonical dual action θ of the dual group $\hat{G}$ on $\mathcal{M} \otimes_\sigma G$ so that $(\mathcal{M} \otimes_\sigma G) \otimes_\theta \hat{G} \cong \mathcal{M} \otimes \mathcal{L}(L^2(G))$, where $\mathcal{L}(L^2(G))$ means, of course, the algebra of all bounded operators on the Hilbert space $L^2(G)$ of all square integrable functions on G with respect to the Haar measure of G.

THEOREM 1. (i) *If H is a closed subgroup of G, then $\mathcal{M} \otimes_\sigma H$ is canonically imbedded in $\mathcal{M} \otimes_\sigma G$ and $\{p \in \hat{G} : \theta_p(x) = x$ for every $x \in \mathcal{M} \otimes_\sigma H\}$ is precisely the annihilator $H^\perp$ of H.*

(ii) *If $\hat{H}$ is a closed subgroup of $\hat{G}$, then the fixed point subalgebra $(\mathcal{M} \otimes_\sigma G)^{\hat{H}}$ of $\mathcal{M} \otimes_\sigma G$ under $\theta(\hat{H})$ is precisely $\mathcal{M} \otimes_\sigma H$, with H the annihilator $\hat{H}^\perp$ of $\hat{H}$ in G.*

We apply this theorem to the structure of von Neumann algebras of type III. In [3], we showed that for a von Neumann algebra $\mathcal{M}$ of type III, there exists uniquely a semifinite von Neumann algebra $\mathcal{M}_0$ equipped with a one parameter automorphism group $\{\theta_t\}$ such that $\mathcal{M} \cong \mathcal{M}_0 \otimes_\theta \boldsymbol{R}$ and the action σ_t of $\boldsymbol{R}$ on $\mathcal{M}$, which is dual to θ, is the modular automorphism group associated with the faithful semifinite normal weight φ which is canonically constructed from a trace τ on $\mathcal{M}_0$ with $\tau \cdot \theta_t = e^t \tau$, $t \in \boldsymbol{R}$. The above theorem implies immediately the following result.

COROLLARY 2. *Imbedding canonically $\mathcal{M}_0$ into $\mathcal{M} = \mathcal{M}_0 \otimes_\theta \boldsymbol{R}$, $\mathcal{M}_0$ is precisely the centralizer of the weight φ.*

COROLLARY 3. *The center $\mathcal{Z}$ of $\mathcal{M}$ is precisely the fixed point subalgebra of the center $\mathcal{Z}_0$ of $\mathcal{M}_0$ under the action $\{\theta_t : t \in \boldsymbol{R}\}$.*

We now consider a factor $\mathcal{M}$ of type III with separable predual $\mathcal{M}_*$. Making use of measure theoretic arguments, which are partly due to

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H. A. Dye, we can show the following:

THEOREM 4. *If* $\mathcal{M}$ *is a factor of type* III *with separable predual* $\mathcal{M}_*$, *then* $\mathcal{M}_0$ *must be of type* II_∞.

Making use of further ergodic theory type arguments, which we owe partly to T. Liggett, we can show the following:

THEOREM 5. *If* $\mathcal{M}$ *is a factor of type* III *with separable predual, for* $\mathcal{M}$ *to be of type* III_1 *in the sense of Connes* (*i.e.*, $S(\mathcal{M}) = \boldsymbol{R}_+$) *it is necessary and sufficient that* $\mathcal{M}_0$ *is a factor of type* II_∞.

COROLLARY 6. *If* $\mathcal{M}$ *is a factor of type* III_1 *with separable predual, then for* $T > 0$ *the crossed product* $\mathcal{M} \otimes_\varphi (T\boldsymbol{Z})$ *of* $\mathcal{M}$ *by the closed subgroup* $T\boldsymbol{Z}$ *of* $\boldsymbol{R}$ *with respect to the modular automorphism group* σ_t^φ *associated with a faithful semifinite normal weight* φ *is a factor of type* III_κ *with* $\kappa = e^{-2\pi/T}$.

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