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REGULAR $O(n)$-MANIFOLDS, SUSPENSION OF KNOTS, AND KNOT PERIODICITY

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1. **Statement of the main results.** It will be convenient for us to define an n-knot to be a smooth, connected, oriented, n-dimensional (closed) submanifold Σ^n of S^{n+2} (oriented). If Σ^n is homeomorphic to S^n, then we call it a *spherical* knot. All manifolds in this note will be oriented and all constructions we consider will induce canonical orientations. This will be understood and not commented upon further.

Let K_n denote the semigroup of isotopy classes of smooth n-knots (S^{n+2}, Σ^n). Our object is to define a homomorphism

$$\omega : K_n \to K_{n+2}$$

which we think is reasonable to call "suspension". This homomorphism ω takes some spherical knots to nonspherical knots and vice-versa. (In fact, $\omega(S^{n+2}, \Sigma^n)$ is just a canonically defined embedding of the cyclic double covering of S^{n+2} branched at Σ^n in S^{n+4}.) However, if we iterate ω twice we obtain the following result:

THEOREM A. *The double suspension $\omega^2 : K_n \to K_{n+4}$ takes homology spherical knots to spherical knots. Moreover, it induces a homomorphism $\omega^2 : C_n \to C_{n+4}$ of (spherical) knot-cobordism groups, which is an isomorphism for $n \neq 1, 3$, an epimorphism for $n = 1$, and a monomorphism onto a subgroup of index two for $n = 3$. Also, ω^2 takes doubly null-cobordant knots to doubly null-cobordant knots.*

That such a homomorphism $C_n \to C_{n+4}$ exists was shown by Levine [7] because of his calculation of these groups, but our result gives the first explicit geometrically defined description of such a homomorphism. (Another, quite different, description has been concurrently and independently discovered by Cappell and Shaneson.)

The first statement in the theorem is an elementary consequence of the construction. The other statements follow from the following stronger facts: Let (S^{n+2}, Σ^n) be an n-knot and let $W^{n+1} \subset S^{n+2}$ be a Seifert surface spanning Σ^n. Then we construct canonically a Seifert surface $\omega(W^{n+1}) \subset S^{n+4}$ for the suspended knot $\omega(S^{n+2}, \Sigma^n)$. We prove that we can regard

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S^{n+4} as the join $S^1 * S^{n+2}$ in such a way that $\omega(W)$ has the suspension $S^0 * W$ as a deformation retract. If W_+ is a displacement of W in the positive normal direction, we can construct a displacement $\omega(W)_+$ of $\omega(W)$ which has $S^0_+ * W_+$ as a deformation retract, where S^0_+ is S^0 rotated slightly in S^1 in the positive direction. Using this, one sees that there are canonical isomorphisms $\tilde{H}_i(W;Z) \approx \tilde{H}_{i+1}(\omega(W);Z)$ and that the linking numbers of classes in $H_*(W)$ with classes in $H_*(W_+)$ equal those of the corresponding classes in $H_*(\omega(W))$ and $H_*(\omega(W)_+)$. This implies the following basic result:

THEOREM B. *Let $(S^{2m+1}, \Sigma^{2m-1})$ be a knot with Seifert surface W^{2m}. Then, with respect to a given basis of $H_m(W^{2m};Z)$ and the induced basis of $H_{m+1}(\omega(W^{2m});Z)$, the Seifert matrix of $\omega(W^{2m}) \subset S^{2m+3}$ is identical to that of $W^{2m} \subset S^{2m+1}$.*

The main part of Theorem A then follows directly from this and the results of Levine [**7**]. (That ω^2 preserves knot-cobordisms is another elementary consequence of the construction.)

A knot $(S^{2m+1}, \Sigma^{2m-1})$ is called *simple* if it has a Seifert surface W^{2m} which is $(m-1)$-connected, and such a W is called simple. Clearly $\omega(W^{2m})$ is simple if W^{2m} is, and thus ω takes simple knots to simple knots.

Using the results of Levine in [**6**], another immediate consequence of Theorem B is the following fact:

THEOREM C. *The double suspension ω^2 gives an isomorphism from the semigroup of isotopy classes of simple spherical $(2m-1)$-knots to that of simple spherical $(2m+3)$-knots for $m \neq 1, 2$. It is a surjection for $m = 1$ and an injection for $m = 2$.*

2. **Regular $O(n)$-manifolds.** In order to describe the construction of the suspension ω, we shall recall some well-known material on (smooth) $O(n)$-manifolds. An $O(n)$-manifold M will be called *regular* if (i) there are only 3 types of orbits: fixed points, spheres $O(n)/O(n-1)$, and the Stiefel manifolds $O(n)/O(n-2)$, (ii) the representation of $O(n)$ at a fixed point is twice the standard representation plus a trivial k-dimensional representation, and (iii) the representation of $O(n-1)$ on the normal space to an orbit of type $O(n)/O(n-1)$ is the standard representation plus a trivial $(k+2)$-dimensional representation. (In fact, (ii) and (iii) follow from (i).) Then $\dim M = 2n + k$ and the orbit space $M/O(n)$ is a topological $(k+3)$-manifold with boundary (the singular orbits) and with the fixed set $M^{O(n)}$ corresponding to a k-dimensional submanifold of the boundary of $M/O(n)$. The orbit space can be given a differentiable structure which is unique up to diffeomorphism preserving $M^{O(n)}$.

A theorem of Jänich [**5**] and Hsiang and Hsiang [**4**] states that for each knot (S^{k+2}, Σ^k) and for $n \geqq 2$ there is a unique (to equivariant diffeomorphism) regular $O(n)$-manifold $M = M^{2n+k}(\Sigma^k)$ such that $(M/O(n), \partial(M/O(n)), M^{O(n)})$ is diffeomorphic to $(D^{k+3}, S^{k+2}, \Sigma^k)$. For a complete exposition of this result and also for the proof in the topological (locally smooth) case, see [**1**]. In case $n = 1, M = M^{2+k}(\Sigma^k)$ is taken to be the cyclic double cover of S^{k+2} branched at Σ^k, so that $(M/O(1), M^{O(1)}) \approx (S^{k+2}, \Sigma^k)$ in this case.

We list some elementary facts.

(1) If Σ^k is a homology sphere, then the $O(2)$-manifold $M^{4+k}(\Sigma^k)$ is a homotopy sphere.

(2) For $O < r < n$ and regarding $O(r) \times O(n-r) \subset O(n)$, the $O(r)$-manifold $M^{2n+k}(\Sigma^k)^{O(n-r)}$ is equivalent to the $O(r)$-manifold $M^{2r+k}(\Sigma^k)$.

(3) $M^{2(n+1)+k}(\Sigma^k)/O(1) \approx S^{2n+k+2}$ and $M^{2(n+r)+k}(\Sigma^k)/O(r) \approx D^{2n+k+3}$ for $r > 1$.

For (1) see [**1**]. Part (2) is an observation that the orbit spaces (and knots) are the same. For (3) one notes that $M^{2(n+1)+k}(\Sigma^k)/O(1)$ is the boundary of $M^{2(n+r)+k}(\Sigma^k)/O(r)$ for $r \geqq 2$ and that the latter are independent of r (use part (2)). But, for r large, the Vietoris mapping theorem applied to the orbit map for the $O(n+r)$-action shows that $M^{2(n+r)+k}(\Sigma^k)$ is highly connected, and, applied to the orbit map for the $O(r)$-action on this, then shows that the $O(r)$-orbit space is contractible. (Also, its boundary $M^{2(n+1)+k}(\Sigma^k)/O(1)$ is simply connected; see [**1**].)

Now

$$M^{2n+k}(\Sigma^k) \approx M^{2(n+1)+k}(\Sigma^k)^{O(1)}$$

is embedded as a codimension two submanifold of

$$M^{2(n+1)+k}(\Sigma^k)/O(1) \approx S^{2n+k+2},$$

and hence is a $(2n+k)$-knot. We define

$$\omega(S^{k+2}, \Sigma^k) = (M^{4+k}(\Sigma^k)/O(1), M^{4+k}(\Sigma^k)^{O(1)})$$

$$\approx (S^{k+4}, M^{2+k}(\Sigma^k)).$$

The above remarks imply easily that the n-fold iteration of ω is just

$$\omega^n(S^{k+2}, \Sigma^k) = (M^{2(n+1)+k}(\Sigma^k)/O(1), M^{2(n+1)+k}(\Sigma^k)^{O(1)})$$

$$\approx (S^{2n+k+2}, M^{2n+k}(\Sigma^k)).$$

We shall not indicate the construction of the Seifert surface $\omega(W)$ from a Seifert surface W cobounding Σ, since it involves a fair amount of explanation. (In fact, the only difficult part of this work was in discovering how to construct such a Seifert surface $\omega(W)$.) Suffice it to say that we investigate the construction of $\omega(S^{k+2}, \Sigma^k)$ closely, using W, and reduce it to a simple cut and paste construction in which $\omega(W)$ and all other facts are relatively evident.

3. **Some consequences.** It is clear that our theorems have some consequences for regular $O(n)$-actions. It follows from Theorem B that suspension preserves the usual knot invariants such as the signature and the Arf-Robertello invariant. For example, the following result is immediate:

COROLLARY. *Let* $(S^{2k+1}, \Sigma^{2k-1})$ *be a knot and let* $n \geqq 2$ *be such that* $n + k$ *is even. Put* $M = M^{2n+2k-1}(\Sigma^{2k-1})$. *If* k *is even, then* M *is a homotopy sphere iff* Σ^{2k-1} *is a homology sphere. If* k *is odd, then* M *is a homotopy sphere iff the cyclic double cover of* S^{2k+1} *branched at* Σ^{2k-1} *is a homology sphere. In these cases,* M *is* $\sigma/8$ *times the standard (Milnor) generator of* bP_{2n+2k}, *where* σ *is the signature of the knot* $(S^{2k+1}, \Sigma^{2k-1})$.

A similar result holds for the Arf-Robertello invariant. This result was proved by Hirzebruch [**3**] and Erle [**2**] in the case $k = 1$.

Another type of application is the following immediate consequence of Theorem C: Suppose that $O(1)$ acts on a manifold M^{4n-1} fixing a homotopy sphere Σ^{4n-3}. Suppose further that the orbit space $M^{4n-1}/O(1)$ is a standard sphere and that the knot $(M^{4n-1}/O(1), \Sigma^{4n-3})$ is *simple*. Then this $O(1)$-action extends to a regular $O(2n-1)$-action. For example, the $O(1)$-action $(z_0, z_1, \ldots, z_{2n}) \longmapsto (-z_0, z_1, \ldots, z_{2n})$ on the Brieskorn manifold

$$z_0^2 + z_1^{a_1} + \cdots + z_{2n}^{a_{2n}} = 0, \qquad \sum |z_i|^2 = 1, \qquad (a_i, a_j) = 1 \text{ for } i \neq j,$$

extends to a regular $O(2n-1)$-action.

A similar application of Theorem A is the following: Suppose that $n \geqq 2$ and $O(2)$ acts regularly on S^{4n+1} fixing a homotopy sphere Σ^{4n-3} and with orbit space D^{4n} (e.g., this is true when the fixed point set of $O(1)$ is simply connected). Then this $O(2)$-action extends to an action on $S^{4n+1} \times I$ fixing an embedded $\Sigma^{4n-3} \times I$ in such a way that the action on the other end extends to a regular $O(2n)$-action on S^{4n+1}.

4. **An example.** It is of interest to ask what $\omega(S^{2n+1}, \Sigma^{2n-1})$ is when $(S^{2n+1}, \Sigma^{2n-1})$ is a Brieskorn knot

$$z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 0, \qquad \sum |z_i|^2 = 1$$

in $S^{2n+1} \subset C^{n+1}$. It is easy to see, in fact, that $\omega(S^{2n+1}, \Sigma^{2n-1})$ is then the Brieskorn knot

$$z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} + z_{n+1}^2 = 0, \qquad \sum |z_i|^2 = 1$$

in $S^{2n+3} \subset C^{n+2}$. This generalizes to (at least) weighted homogeneous polynomials.

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