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COMPACT HILBERT CUBE MANIFOLDS AND THE INVARIANCE OF WHITEHEAD TORSION

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ABSTRACT. In this note we prove that every compact metric manifold modeled on the Hilbert cube Q is homeomorphic to $|K| \times Q$, for some finite simplicial complex K. We also announce an affirmative answer to the question concerning the topological invariance of Whitehead torsion for compact, connected CW-complexes. As a corollary of this latter result it follows that two compact Hilbert cube manifolds are homeomorphic iff their associated polyhedra (in the sense above) have the same simple homotopy type.

1. **Introduction.** In this note we announce some recent results on infinite-dimensional manifolds which imply, among other things, the topological invariance of Whitehead torsion for compact connected CW-complexes.

A *Hilbert cube manifold* (or Q-manifold) is a separable metric space which has an open cover by sets which are homeomorphic to open subsets of the Hilbert cube Q. We say that a Q-manifold X can be *triangulated* (or is *triangulable*) provided that X is homeomorphic ($\cong$) to $|K| \times Q$, for some countable locally-finite simplicial complex K. In [**5**] it was shown that (1) any open subset of Q is triangulable, and (2) if X is any Q-manifold, then $X \times [0, 1)$ is openly embeddable in Q and thus is triangulable (where $[0, 1)$ is the half-open interval). We refer the reader to [**4**] for a list of earlier results on Q-manifolds. In this note, based on results in [**6**], we prove that every compact Q-manifold can be triangulated. The triangulation of non-compact Q-manifolds is much more delicate and is expected to be the subject of a future paper.

TRIANGULATION THEOREM. *Every compact Q-manifold can be triangulated.*

Using a result of West [**13**] it follows that if K is any finite simplicial complex, then $|K| \times Q \cong M \times Q$, for some finite-dimensional combinatorial manifold M. In this sense it follows that all compact Q-manifolds can be *combinatorially* triangulated.

COROLLARY 1. *Every compact Q-manifold can be combinatorially triangulated.*

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This result contrasts sharply with the corresponding finite-dimensional situation, as there exist compact finite-dimensional manifolds which cannot be combinatorially triangulated [**8**].

In [**13**] West proved that if K is any finite complex, simplicial or CW, then $|K| \times Q$ is a Q-manifold. Combining this result with the *Triangulation Theorem* we have the following result on finiteness of homotopy types.

COROLLARY 2. *If X is a compact metric space which is locally triangulable, then X has the homotopy type of a finite complex.*

In particular X might be a compact n-manifold. Thus Corollary 2 strengthens the result of Kirby-Siebenmann on the finiteness of homotopy types of compact n-manifolds [**9**], but it is questionable whether this gives a simpler proof. It also sheds some light on the more general open question concerning the finiteness of homotopy types of compact ANR's [**12**].

The following result gives a topological characterization of simple homotopy types of finite CW-complexes in terms of homeomorphisms on Q-manifolds. We refer the reader to [**7**] for a proof.

CHARACTERIZATION OF SIMPLE HOMOTOPY TYPES. *Let K, L be finite, connected* CW*-complexes and let $f : |K| \to |L|$ be a map. Then f is a simple homotopy equivalence iff the map*

$$f \times \mathrm{id} : |K| \times Q \to |L| \times Q$$

is homotopic to a homomorphism of $|K| \times Q$ onto $|L| \times Q$.

We remark that the "only if" part of this theorem is just West's theorem on simple homotopy types [**13**]; i.e., if $f : |K| \to |L|$ is a simple homotopy equivalence, then $f \times \mathrm{id} : |K| \times Q \to |L| \times Q$ is homotopic to a homeomorphism. As an immediate corollary of the above theorem we answer affirmatively the question on the topological invariance of Whitehead torsion [**11**, p. 378].

Waldhausen is reported to have an earlier and completely different proof of this Whitehead invariance problem.

COROLLARY 3. *If K, L are finite, connected* CW*-complexes and $f : |K| \to |L|$ is a homeomorphism, then f is a simple homotopy equivalence.*

We can also easily use the above *Characterization* to classify homeomorphic compact, connected Q-manifolds by simple homotopy type.

COROLLARY 4. *Let X, Y be compact, connected Q-manifolds and let*

$$X \cong |K| \times Q, \qquad Y \cong |L| \times Q$$

be any triangulations. Then $X \cong Y$ iff $|K|$ and $|L|$ have the same simple homotopy type.

Our proof we give here of the *Triangulation Theorem* and the proof of the *Characterization of Simple Homotopy Types*, which is given in [**7**], rely mainly on the work in [**6**]. For our purposes here we use the following result from [**6**].

For notation R^n denotes Euclidean n-space, B^n denotes the standard n-ball of radius 1, $\mathrm{Int}(B^n)$ denotes the interior of B^n, and S^{n-1} denotes the boundary.

STRAIGHTENING LEMMA. *If X is a triangulated Q-manifold and $h: R^n \times Q \to X$ is an open embedding, for $n \geqq 2$, then $X \backslash h(\mathrm{Int}(B^n) \times Q)$ is a triangulated Q-manifold.*

The proof of the *Straightening Lemma* given in [**6**] uses a considerable amount of infinite-dimensional machinery—influenced by finite-dimensional techniques. Some of the techniques used are infinite-dimensional surgery and infinite-dimensional handle straightening. These are just infinite-dimensional versions of some finite-dimensional techniques which were used in [**9**] for the recent triangulation results concerning n-manifolds.

Doing surgery on Q-manifolds is not nearly as difficult as it is on n-manifolds. In particular, the delicate inductive process of exchanging handles can always be done in two steps. Poincaré duality and transversality are never used. For straightening handles in Q-manifolds homeomorphisms on $T^n \times Q$ (where T^n represents the n-torus) are used, much in the same way that torus homeomorphisms are used in finite-dimensional handle straightening. Also the theorem of West on simple homotopy type plays the role of the s-cobordism theorem.

In §2 we briefly describe some infinite-dimensional results which will be needed. Then in §3 we use these results, along with the *Straightening Lemma*, to prove the *Triangulation Theorem*. We remark that no prior experience with infinite-dimensional topology is needed for reading this note.

2. **Infinite-dimensional preliminaries.** We view Q as the countable infinite product of closed intervals $[-1, 1]$ and we let 0 represent the point $(0, 0, \ldots)$ of Q. Most basic is Anderson's notion of Property Z [**1**]. A closed subset A of a space X is said to be a *Z-set in X* provided that for each nonnull and homotopically trivial open subset U of X, $U \backslash A$ is also nonnull and homotopically trivial. A map $f: X \to Y$ (i.e., a continuous function) is said to be a *Z-embedding* provided that f is a homeomorphism of X onto a Z-set in Y. We now state two properties of Z-sets in Q-manifolds which will be needed in §3. For details see [**2**] and [**5**].

APPROXIMATION LEMMA. *Let A be a compact metric space, X be a Q-manifold, and let $f: A \to X$ be a map. Then f is homotopic to a Z-embedding $g: A \to X$. In fact, g can be chosen arbitrarily close to f.*

COLLARING LEMMA. *Let K be a finite complex, X be a Q-manifold, and let $f:|K| \times Q \to X$ be a Z-embedding. Then there exists an open embedding $g:|K| \times Q \times [0, 1) \to X$ such that $g(k, q, 0) = f(k, q)$, for all $(k, q) \in |K| \times Q$.*

We will also need the following result on the characterization of Q. For details see [**5**].

CHARACTERIZATION OF Q. *If X is a compact contractible Q-manifold, then $X \cong Q$.*

3. **Proof of the Triangulation Theorem.** Without loss of generality, let us consider a compact connected Q-manifold X which we describe to triangulate. Since X is a compact ANR (metric), it follows that X is dominated by a finite complex (see [**3**, p. 106]). Thus $\pi_1(X)$ and $H_*(X)$ are finitely generated, where we use singular homology with integral coefficients. By a standard process, all of the homotopy groups of X can be killed by attaching a finite number of n-cells to X, for $n \geqq 2$. One uses the fact that $\pi_1(X)$ is finitely generated to kill $\pi_1(X)$, and after this the Hurewicz isomorphism theorem is used to inductively kill the higher homotopy groups. Since $H_*(X)$ is finitely generated the process terminates after a finite number of cell attachments. We omit the details. Thus we obtain a finite sequence $X = X_0, X_1, \ldots, X_p$ of spaces such that each X_i is obtained by attaching some n-cell to X_{i-1}, and $\pi_n(X_p) = 0$ for all n. Since X_p is an ANR, it must be contractible (see [**3**, p. 124]). Of course the spaces X_i, $1 \leqq i \leqq p$, are not Q-manifolds. We are going to modify the above procedure to obtain a sequence $X = X'_0, X'_1, \ldots, X'_p$ of compact Q-manifolds such that each X'_i has the homotopy type of X_i. We construct this sequence inductively.

For each j, $0 \leqq j \leqq p$, let S_j be the following statement: There exists a sequence $X = X'_0, X'_1, \ldots, X'_j$ of compact Q-manifolds such that (1) for $0 \leqq i \leqq j$, X'_i has the homotopy type of X_i, and (2) for $1 \leqq i \leqq j$, if X'_i is triangulable, then so is X'_{i-1}. It is clear that S_0 is true. Thus assume that S_j is true, for some $j < n$, and let $X = X'_0, X'_1, \ldots, X'_j$ be the corresponding sequence of Q-manifolds satisfying (1) and (2). Let $\alpha: X_j \to X'_j$ be a homotopy equivalence and let $f: S^{n-1} \to X_j$ be the attaching map used to construct X_{j+1}; i.e., $X_{j+1} = X_j \cup_f B^n$. Define $f': S^{n-1} \times Q \to X'_j$ by $f'(x, q) = \alpha f(x)$, for all $(x, q) \in S^{n-1} \times Q$, and using the *Approximation Lemma* let $g: S^{n-1} \times Q \to X'_j$ be a Z-embedding which is homotopic to f'. We define $X'_{j+1} = X'_j \cup_g (B^n \times Q)$, the space formed by attaching $B^n \times Q$ to X'_j. It easily follows from the *Collaring Lemma* that X'_{j+1} is a Q-manifold. In fact, there exists an open embedding $h: R^n \times Q \to X'_{j+1}$ such that $X'_j \cong X'_{j+1} \backslash h(\mathrm{Int}(B^n) \times Q)$. Applying the *Straightening Lemma* it follows that if X'_{j+1} is triangulable, then so is X'_j. All we need to do now is check that X'_{j+1} and X_{j+1} have the same homotopy type. Let $g' : S^{n-1} \to X'_j$ be defined by $g'(x) = g(x, 0)$, for all $x \in S^{n-1}$. Since Q contracts to $0 \in Q$, it follows that X'_{j+1} has the homotopy of $X'_j \cup_{g'} B^n$. But g' is homotopic to $\alpha f : S^{n-1} \to X'_j$. Using Theorem 2.3 on p. 120 of [**10**], it follows that $X'_j \cup_{g'} B^n$ has the homotopy type of X_{j+1}. Thus S_{j+1} is true.

Since S_p is true we have a sequence $X = X'_0, X'_1, \ldots, X'_p$ of compact Q-manifolds such that X'_p is contractible and if X'_i is triangulable, then so is X'_{i-1}, for $1 \leqq i \leqq p$. Using the *Characterization of* Q it follows that $X'_p \cong Q$, which is triangulable. Then inductively working our way back down to X it follows that X is triangulable.

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