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BOUNDARY VALUE PROBLEMS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH RAPIDLY INCREASING COEFFICIENTS

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1. **Introduction.** The purpose of this note is to present a general existence theorem for variational boundary value problems for quasilinear elliptic operators in divergence form:

$$A(u) = \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \ldots, \nabla^m u), \tag{1}$$

in the case where the coefficients A_α do not have polynomial growth in u and its derivatives. The crucial points in the treatment of rapidly (or slowly) increasing A_α's are that the Banach spaces in which the problems are appropriately formulated are nonreflexive and that the corresponding operators are not bounded nor everywhere defined and do not generally satisfy a global a priori bound. This existence theorem is based upon an extension of the theory of not everywhere defined unbounded pseudo-monotone mappings (Browder [5], [6], Browder-Hess [7]) to the context of complementary systems.

Detailed proofs will appear elsewhere.

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2. **Main results.** We will use the following notations. If $\xi = \{\xi_\alpha : |\alpha| \leqq m\} \in R^{s_m}$ is a m-jet, then $\zeta = \{\xi_\alpha : |\alpha| = m\} \in R^{s'_m}$ denotes its top order part and $\eta = \{\xi_\alpha : |\alpha| < m\} \in R^{s_{m-1}}$ its lower order part; for u a derivable function, $\xi(u)$ denotes $\{D^\alpha u : |\alpha| \leqq m\}$. The Orlicz space [11] on $\Omega \subset R^n$ corresponding to an N-function M is denoted by $L_M(\Omega)$ and the closure in $L_M(\Omega)$ of the simple functions in Ω by $E_M(\Omega)$. The Sobolev space of functions u such that u and its distribution derivatives up to order m lie in $L_M(\Omega)$ [$E_M(\Omega)$] is denoted by $W^m L_M(\Omega)$ [$W^m E_M(\Omega)$]; these spaces are identified to subspaces of the product $\prod_{|\alpha| \leqq m} L_M(\Omega) \equiv \prod L_M$. $\overline{M}$ [M^{-1}] denotes the function conjugate [reciprocal] to M and $N \ll M$ means that, for each $\varepsilon > 0$, $M(\varepsilon t)/N(t) \to +\infty$ as $t \to +\infty$.

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Let Ω be a bounded open subset of R^n such that the Sobolev imbedding theorem holds on Ω. The basic conditions we impose on the coefficients A_α are the following:

(i) Each $A_\alpha(x, \xi)$ is a real-valued function defined on $\Omega \times R^{s_m}$ which is measurable in x for fixed ξ and continuous in ξ for fixed x.

(ii) There exist N-functions M and N with $N \ll M$, $a(x) \in E_{\overline{M}}(\Omega)$ and $b, c \in R^+$ such that

$$\text{if } |\alpha| = m: \ |A_\alpha(x, \xi)| \leqq a(x) + b \sum_{|\beta|=m} \overline{M}^{-1}M(c\xi_\beta) + b \sum_{|\beta|<m} \overline{N}^{-1}M(c\xi_\beta),$$

$$\text{if } |\alpha| < m: \ |A_\alpha(x, \xi)| \leqq a(x) + b \sum_{|\beta|=m} \overline{M}^{-1}N(c\xi_\beta) + b \sum_{|\beta|<m} \overline{M}^{-1}M(c\xi_\beta),$$

for all x in Ω and ξ in R^{s_m}. (This assumption can be weakened using the Sobolev imbedding theorem of [9].)

(iii) For each x in Ω, η in $R^{s_{m-1}}$, ζ and ζ' in $R^{s'_m}$ with $\zeta \neq \zeta'$,

$$\sum_{|\alpha|=m} (A_\alpha(x, \zeta, \eta) - A_\alpha(x, \zeta', \eta))(\zeta_\alpha - \zeta'_\alpha) > 0;$$

for each x in Ω, ζ' and ζ'' in $R^{s'_m}$,

$$\sum_{|\alpha|=m} (A_\alpha(x, \zeta, \eta) - \zeta'_\alpha)(\zeta_\alpha - \zeta''_\alpha) \to +\infty$$

as $|\zeta| \to +\infty$ in $R^{s'_m}$, uniformly for bounded η in $R^{s_{m-1}}$.

Let Y be a $\sigma(\prod L_M, \prod E_{\overline{M}})$ closed subspace of $W^m L_M(\Omega)$ on which we impose the condition

$$\text{(iv)} \qquad Y = \sigma(\prod L_M, \prod L_{\overline{M}}) \text{ cl } Y_0$$

where $Y_0 = Y \cap W^m E_M(\Omega)$; here M is the N-function involved in condition (ii). Let $f \in Y_0^*$. The variational boundary value problem (VBVP) for $A(u) = f$ with respect to Y asks for an element u in Y such that $A_\alpha(\xi(u)) \in L_{\overline{M}}(\Omega)$ for all α and

$$a(u, v) \equiv \sum_{|\alpha| \leqq m} \int_\Omega A_\alpha(\xi(u)) D^\alpha v \, dx = f(v),$$

for all v in Y_0.

More generally we consider a one-parameter family of operators

$$A_t(u) = \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \ldots, \nabla^m u, t), \tag{2}$$

where $t \in [0, 1]$. The coefficients $A_\alpha(x, \xi, t)$ are assumed to satisfy (i), (ii), (iii) for each t; moreover it is assumed that they are continuous in (ξ, t)

for fixed x, that the functions $M, N, a(x)$ and the constants b, c of (ii) can be chosen independently of t and that the convergence in the second part of (iii) is uniform in t. Briefly we will say that (i), (ii), (iii) are satisfied uniformly in t.

THEOREM 1. *Let* $\{A_t : t \in [0, 1]\}$ *be a one-parameter family of operators of the form* (2) *satisfying* (i), (ii), (iii) *uniformly in* t. *Let* Y *be a* $\sigma(\prod L_M, \prod E_{\overline{M}})$ *closed subspace of* $W^m L_M(\Omega)$ *satisfying* (iv). *Suppose that* A_1 *is odd and that for each* $\sigma(\prod E_M, \prod E_{\overline{M}})$ *continuous linear form* f *on* Y_0 *there exist a constant* K *and a neighbourhood* $\mathcal{N}$ *of* f *in* Y_0^* *such that for any* g *in* $\mathcal{N}$, *any* t *in* $[0, 1]$ *and any solution* u *of the VBVP for* $A_t(u) = g$ *with respect to* Y, $\|u\| \leqq K$. *Then, for each* t *in* $[0, 1]$ *and each* $\sigma(\prod E_M, \prod E_{\overline{M}})$ *continuous linear form* f *on* Y_0, *the VBVP for* $A_t(u) = f$ *with respect to* Y *has at least one solution.*

Simple examples show that the above VBVP may have no solution if f is arbitrary in Y_0^*. Assumption (iv) is satisfied for instance by $W^m L_M(\Omega)$ or $W_0^m L_M(\Omega) \equiv \sigma(\prod L_M, \prod E_{\overline{M}}) \operatorname{cl} \mathcal{D}(\Omega)$ if Ω has the segment property. Theorem 1 can be applied in particular to the operator

$$\sum_{|\alpha| = m} (-1)^{|\alpha|} D^\alpha(p(D^\alpha u)) + \text{lower order terms},$$

where $p : R \to R$ is any strictly increasing odd continuous function with $p(+\infty) = +\infty$ and where the lower order terms satisfy some growth condition involving p and a sign condition.

The following result, in which the Dirichlet form $a(u, v)$ is assumed to be coercive, can be derived as in [**4**] from Theorem 1.

THEOREM 2. *Let* A *be an operator of the form* (1) *satisfying* (i), (ii), (iii). *Let* Y *be a* $\sigma(\prod L_M, \prod E_{\overline{M}})$ *closed subspace of* $W^m L_M(\Omega)$ *satisfying* (iv). *Suppose that* $a(u, u)/\|u\| \to +\infty$ *as* $\|u\| \to +\infty$ *in* Y *with* $A_\alpha(\xi(u)) \in L_{\overline{M}}(\Omega)$ *for all* α. *Then, for each* $\sigma(\prod E_M, \prod E_{\overline{M}})$ *continuous linear form* f *on* Y_0, *the VBVP for* $A(u) = f$ *with respect to* Y *has at least one solution.*

Existence theorems for problems of this type were first obtained by Višik [**14**], [**15**] using a priori estimates on $(m + 1)$st derivatives. In the case of coefficients with polynomial growth, monotonicity methods were first applied to these problems by Browder [**2**]; basic improvements of Browder's original results were given by Leray-Lions [**12**] who introduced condition (iii) and proved an analogue of Theorem 2, and by Browder [**4**] who considered noncoercive problems and proved an analogue of Theorem 1. In the case of rapidly increasing coefficients, Donaldson [**8**] (see also [**10**]) obtained the simpler version of Theorem 2 corresponding to the situation where the A_α's satisfy a monotonicity condition with respect to all the derivatives of u and where $\overline{M}$ satisfies the Δ_2 condition. Recently Browder [5] considered equations with top order terms of polynomial growth but lower order terms of rapid growth.

3. **Abstract results.** The proof of Theorem 1 rests upon general results on nonlinear operators of monotone type in nonreflexive Banach spaces.

DEFINITION 1. Let E and F be Banach spaces in duality with respect to a continuous pairing $\langle , \rangle$ and let E_0 and F_0 be subspaces of E and F respectively. Then $(E, E_0; F, F_0)$ is called a complementary system if, by means of $\langle , \rangle$, E_0^* can be identified to F and F_0^* to E.

For instance $(\prod L_M, \prod E_M; \prod L_{\overline{M}}, \prod E_{\overline{M}})$ is a complementary system, and if we take a $\sigma(\prod L_M, \prod E_{\overline{M}})$ closed subspace Y of $\prod L_M$ and successively define $Y_0 = Y \cap \prod E_M$, $Z = \prod L_{\overline{M}}/Y_0^{\perp}$ and $Z_0 = \{f + Y_0^{\perp} : f \in \prod E_{\overline{M}}\}$, then the pairing between $\prod L_M$ and $\prod L_{\overline{M}}$ induces a pairing between Y and Z iff Y satisfies (iv), in which case $(Y, Y_0; Z, Z_0)$ is a complementary system; we will refer to it as the complementary system generated by Y. An (equivalent) norm $\| \ \|_E$ on E will be called admissible if it is lower semicontinuous for $\sigma(E, F_0)$ and satisfies $\langle y, z\rangle \leqq \|y\|_E \, \|z\|_F$ for all y in E and z in F, where $\| \ \|_F$ is obtained by first restricting $\| \ \|_E$ to E_0 and then taking the dual norm.

DEFINITION 2. Let $(Y, Y_0; Z, Z_0)$ be a complementary system and let V be a dense subspace of Y_0. A one-parameter family of mappings T_t of $D(T_t) \subset Y$ into Z, $t \in [0, 1]$, is said to define a pseudo-monotone homotopy with respect to V if (a) $V \subset D(T_t)$ for each t and T is finitely continuous from $[0, 1] \times V$ to the $\sigma(Z, V)$ topology of Z, (b) for any sequences u_i in V and t_i in $[0, 1]$ such that $u_i \to u \in Y$ for $\sigma(Y, Z_0)$, $t_i \to t$, $T_{t_i}(u_i) \to v \in Z$ for $\sigma(Z, V)$ and $\limsup \langle u_i, T_{t_i}(u_i)\rangle \leqq \langle u, v\rangle$, it follows that $u \in D(T_t)$, $T_t(u) = v$ and $\lim \langle u_i, T_{t_i}(u_i)\rangle = \langle u, v\rangle$. In particular, each mapping T_t is pseudo-monotone with respect to V (where the latter is defined in a similar way).

The following two theorems, together with a geometric result of Rao [13], imply Theorem 1. They extend corresponding results by Browder [3], [5].

THEOREM 3. *Let $\{A_t : t \in [0, 1]\}$ be a one-parameter family of operators of the form* (2) *satisfying* (i), (ii), (iii) *uniformly in t. Let Y be a $\sigma(\prod L_M, \prod E_{\overline{M}})$ closed subspace of $W^m L_M(\Omega)$ satisfying* (iv) *and let $(Y, Y_0; Z, Z_0)$ be the complementary system generated by Y. For each t, let T_t be the mapping of $D(T_t) = \{u \in Y : A_\alpha(\xi(u), t) \in L_{\overline{M}}(\Omega)$ for all $\alpha\}$ into Z defined by $\langle v, T_t(u)\rangle = a_t(u, v)$ for all $v \in Y_0$, where $a_t(u, v)$ is the Dirichlet form associated with A_t. Then $\{T_t : t \in [0, 1]\}$ defines a pseudo-monotone homotopy with respect to any dense subspace V of Y_0.*

THEOREM 4. *Let $(Y, Y_0; Z, Z_0)$ be a complementary system and consider a one-parameter family of mappings T_t of $D(T_t) \subset Y$ into Z, $t \in [0, 1]$, which defines a pseudo-monotone homotopy with respect to any dense subspace V of a dense subspace V' of Y_0. Suppose that T_1 is odd on V' outside some ball and that for each z in Z_0 there exists a neighbourhood $\mathcal{N}$ of z in Z such that $\bigcup \{T_t^{-1}(\mathcal{N}): t \in [0, 1]\}$ is bounded in Y. Suppose that Y_0 and Z_0 are separable and that Y admits an equivalent admissible norm whose restriction to Y_0 is Gâteaux differentiable. Then for each t in $[0, 1]$, the range of T_t contains Z_0.*

Pseudo-monotonicity was introduced by Brézis [1]; the extension of Brézis' original results to non everywhere defined unbounded mappings in reflexive Banach spaces was carried out by Browder-Hess [7], with applications in Browder [5] to partial differential equations. The concept of pseudo-monotone homotopy is due to Browder [5], [6]. Complementary systems were defined in [9].

ADDED IN PROOF. Theorem 2 also includes the result announced recently by A. Fougères (C. R. Acad. Sci. Paris, February 1972) where $\overline{M}$ is required to satisfy the Δ_2 condition.

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