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PERTURBATION OF EMBEDDED EIGENVALUES[1]

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In [**4**] a Weinstein-Aronszajn multiplicity theory for embedded eigenvalues arising from a certain type of "resonance" was developed. The results announced here continue the work of [**4**], and generalize results of [**2**] and [**3**] to embedded eigenvalues of arbitrary finite multiplicity m, and to perturbations of infinite rank. In particular, we are able to discuss certain operators of quantum mechanics. A notable feature of the case $m > 1$ is the appearance of Puiseux series for the resonances, in analogy to their appearance in the perturbation theory of *isolated* eigenvalues of *nonselfadjoint* operators [**6**, Chapters 2 and 7].

1. **Puiseux series for resonances.** Let T be a selfadjoint operator on a separable Hilbert space $\mathscr{H}$, with resolvent $G(z) = (T - z)^{-1}$, and let λ_0 be a point eigenvalue of T of finite multiplicity m. Denote by P the orthogonal projection on $\ker(T - \lambda_0)$. Let A and B be bounded commuting selfadjoint operators on $\mathscr{H}$, and define

$$H(\kappa) = T + \kappa AB.$$

For real κ, $H(\kappa)$ is selfadjoint and we define $R(z, \kappa) = (H(\kappa) - z)^{-1}$. Let Ω be a neighborhood λ_0 in the complex plane, and assume that the operator $Q(z) = AG(z)B$ is bounded and has meromorphic continuations $Q^{\pm}(z)$ from $\Omega^{\pm} = \{z \in \Omega\colon \operatorname{Im} z > 0\}$ to Ω. There is then a simple pole of $Q^+(z)$ at λ_0 with residue APB. The functions $Q^+(z)$ and $Q^-(z)$ will not agree on Ω if the eigenvalue λ_0 is embedded in the continuous spectrum of T. The operator

$$Q_1(z, \kappa) = AR(z, \kappa)B$$

also has meromorphic continuations $Q_1^{\pm}(z, \kappa)$ given by

$$I - \kappa Q_1(z, \kappa) = [I + \kappa Q(z, \kappa)]^{-1}.$$

It is the poles of $Q_1^+(z, \kappa)$ that we refer to as the *resonances* of this perturbation problem.

The following was proved in [**4**, §5].

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THEOREM 1. *There is an analytic function* $\Delta(z, \kappa)$ *on a polydisc* $\{(z, \kappa): |z - \lambda_0| < \delta_1, |\kappa| < \delta_2\}$ *such that*

(a) *for* $|\kappa| < \delta_2$, $\Delta(z, \kappa)$ *has exactly* m *zeros* $z_1(\kappa), \dots, z_m(\kappa)$ *(repeated according to multiplicity) in* $|z - \lambda_0| < \delta_1$, *which are precisely the poles of* $Q_1^+(z, \kappa)$ *in* $|z - \lambda_0| < \delta_2$. *For* $\kappa = 0$, $z_j(0) = \lambda_0$ $(j = 1, \dots, m)$.

(b) *If for some real* κ, $z_j(\kappa)$ *is real, then* $z_j(\kappa)$ *is an eigenvalue of* $H(\kappa)$ *of multiplicity equal to the multiplicity* $m_j(\kappa)$ *of* $z_j(\kappa)$ *as a zero of* $\Delta(z, \kappa)$.

The next theorem discusses the dependence of $z_j(\kappa)$ on κ.

THEOREM 2. *The resonances* $z_1(\kappa), \dots, z_m(\kappa)$ *may be labeled so that each* $z_j(\kappa)$ *has a Puiseux series expansion in* κ. *If*

$$z_j(\kappa) = \lambda_0 + \alpha_1 \omega^j \kappa^{1/p} + \alpha_2 \omega^{2j} \kappa^{2/p} + \cdots \qquad (j = 1, \dots, p)$$

is a given Puiseux cycle of resonances, where ω *is a primitive pth root of unity, then either the series has the form*

$$z_j(\kappa) = \lambda_0 + \alpha_p \kappa + \cdots + \alpha_{2np} \kappa^{2n} + \alpha_{2np+1} \omega^j \kappa^{(2n+1)/p} + \cdots$$

where $\lambda_0, \alpha_p, \dots, \alpha_{(2n-1)p}$ *are real and* $\operatorname{Im} \alpha_{2np} > 0$; *or* $p = 1$ *and all the coefficients* α_n *are real.*

Moreover, the multiplicity $m_j(\kappa)$ *is independent of* κ *for* $\kappa \neq 0$ *and sufficiently small, and is the same for each element* $z_j(\kappa)$ *of a given Puiseux cycle.*

In particular, if $z_j(\kappa)$ belongs to a Puiseux cycle with $p \geqq 2$, then $z_j(\kappa)$ is not real for all sufficiently small $\kappa \neq 0$. Thus any actual embedded eigenvalues of $H(\kappa)$ are analytic.

COROLLARY. *For real* $\kappa \neq 0$ *sufficiently small, the multiplicity of point eigenvalues in the interval* $(\lambda_0 - \delta_1, \lambda_0 + \delta_1)$ *is independent of* κ. *If* $z_j(\kappa)$ *is real for all sufficiently small* κ, *then* $z_j(\kappa)$ *is analytic in* κ.

An example can be given in which an eigenvalue of multiplicity $m = 2$ gives rise to a nonanalytic Puiseux expansion in powers of $\kappa^{1/2}$. The perturbation AB in this example has rank 4.

Let $\phi_1, \dots, \phi_m$ be an orthonormal basis of $P\mathscr{H}$ in which the selfadjoint operator $PABP$ on $P\mathscr{H}$ is diagonal.

THEOREM 3. *If the eigenvalues* $A_1, \dots, A_m$ *of the operator* $PABP$ *on* $P\mathscr{H}$ *are all distinct, then* $z_j(\kappa)$ $(j = 1, \dots, m)$ *are analytic in* κ, *and*

$$z_j(\kappa) = \lambda_0 + \kappa \lambda_j + \kappa^2 (Q_c^+(\lambda_0) A\phi_j, B\phi_j) + O(\kappa^3) \qquad (j = 1, \dots, m),$$

where

$$Q_c^+(z) = Q^+(z) - (\lambda_0 - z)^{-1} APB$$

is the analytic continuation of $AG(z)P_cB$ *to a neighborhood of* λ_0, *and* $P_c = I - P$.

For real κ, this implies that to a first approximation

$$-\operatorname{Im} z_j(\kappa) = \pi\kappa^2[d(E(\lambda)P_cV\phi_j, V\phi_j)/d\lambda]_{\lambda=\lambda_0} \qquad (j = 1, \ldots, m)$$

where $V = AB$ and $T = \int \lambda\, dE(\lambda)$. This result is known to physicists as *Fermi's Golden Rule*.

2. **Spectral concentration.** The proof of the following result on spectral concentration involves a grouping of the resonances into "clusters" in such a way that each cluster behaves asymptotically like a single *simple* pole of $Q_1^+(z, \kappa)$ (cf. the construction in [**5**]).

THEOREM 4. *For* $j = 1, \ldots, m$ *and* κ *real, choose* $\delta_j(\kappa)$ *such that* $\delta_j(\kappa) = o(1)$ *and* $\operatorname{Im} z_j(\kappa) = o(\delta_j(\kappa))$ *as* $\kappa \to 0$. *Let*

$$S(\kappa) = \bigcup_{j=1}^{m} \{t : \operatorname{Re} z_j(\kappa) - \delta_j(\kappa) < t < \operatorname{Re} z_j(\kappa) + \delta_j(\kappa)\}.$$

If $H(\kappa) = \int \lambda\, dE_\kappa(\lambda)$, *then*

$$P = \operatorname*{st-lim}_{\kappa\to 0} \int_{S(\kappa)} dE_\kappa(\lambda).$$

3. **The Auger phenomenon.** The results above may be applied to the Schroedinger operator

$$(*) \qquad H(\kappa) = -\Delta + V_1(x) + V_2(y) + \kappa V_3(x - y)$$

acting on functions $u(x, y) \in L_2(R_6)$, where $x, y \in R_3$ and Δ is the 6-dimensional Laplacian. The functions $V_i(x)$ are assumed to be measurable on R_6 and to satisfy

$$|V_i(x)| \leqq ce^{-\alpha|x|} \qquad (i = 1, 2, 3).$$

One may describe $H(\kappa)$ as the Hamiltonian of a helium-like atom (with short range potentials), and the phenomenon discussed here is then analogous to the Auger effect.

The boundedness assumption on A and B can be weakened sufficiently to include potentials $V_i(x)$ which are locally L_2, and, in particular, the Yukawa potential.

Very recently, Simon [7] has announced an analyticity result, for simple multiplicity ($m = 1$), which applies to operators of the type $(*)$ where $V_i(x)$ are analytic functions of $|x|$, and includes, in particular, the Coulomb case. His work is based on results of Balslev and Combes [**1**].

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