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ON HAEFLIGER'S CLASSIFYING SPACE. I

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Haefliger has defined a classifying space $B\Gamma$ for any topological groupoid Γ (cf. [1] and [2]). In this note, we announce some results concerning the homology of $B\Gamma_1^r$, where Γ_1^r is the pseudogroup of local C^r diffeomorphisms of the real line $\boldsymbol{R}$, and r is a nonnegative integer or ∞.

The differential induces a map $\Gamma_1^r \to Gl(1, \boldsymbol{R})$ and hence a map $B\Gamma_1^r \to BGl(1, \boldsymbol{R}) = B\boldsymbol{Z}_2$ (cf. [1]). We denote the fiber of this map (in the sense of homotopy theory) by $F\Gamma_1^r$. Clearly $F\Gamma_1^r$ is a 2-fold cover of $B\Gamma_1^r$. Haefliger has shown that it is simply connected [2]. The main result announced in this paper is the construction of a chain complex $\beta(\beta G^r)$ whose homology is the same as the homology of $F\Gamma_1^r$. Here G^r denotes the group of C^r diffeomorphisms of $\boldsymbol{R}$ with compact support, βG^r the chain complex obtained by applying the bar construction to the integral group ring $\boldsymbol{Z}G^r$, and $\beta(\beta G^r)$ a chain complex obtained from βG^r by a procedure analogous to the bar construction.

From our main result, we obtain two spectral sequences relating the homology of $F\Gamma_1^r$ and the homology of the discrete group G^r. From either spectral sequence and the fact that $F\Gamma_1^r$ is simply connected, it follows that $F\Gamma_1^r$ is contractible if and only if the homology of G^r vanishes in all positive dimensions. For $r=0$, it is known that this homology vanishes [3]; for $r>0$, this is unknown. (Added in proof. For $r \geqq 2$, it is now known that $H_3(F\Gamma_1^r) \neq 0$. Cf. [5].)

1. **A modified bar construction.** We will refer to an n-tuple $(g_1, \cdots, g_n)$ of elements of G^r as an n-cell, and an n-cell for which at least one g_i is the identity as a *degenerate* n-cell. We may identify βG^r with the chain complex whose n-chains are the members of the free abelian group generated by the n-cells, modulo the subgroup generated by the degenerate n-cells, and whose boundary is given by

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$$\partial(g_1, \cdots, g_n) = (g_2, \cdots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \cdots, g_i g_{i+1}, \cdots, g_n) + (-1)^n (g_1, \cdots, g_{n-1}).$$

(We make the convention that there is one 0-cell and that the boundary of any 1-chain is 0.) It is well known that the homology of βG^r is the same as the homology of G^r, considered as a discrete group.

Given an n-cell $c^n = (g_1, \cdots, g_n)$, we define its *interval* $I(c^n)$ to be the smallest interval containing $\operatorname{supp} g_1 \cup \cdots \cup \operatorname{supp} g_n$. Given two intervals I_1 and I_2, we write $I_1 < I_2$ if and only if $s \in I_1$, $t \in I_2 \Rightarrow s < t$. If $c^m = (g_1, \cdots, g_m)$ and $c^n = (g_{m+1}, \cdots, g_{m+n})$ are cells, we define

$$\mu(c^m, c^n) = \sum_{\pi} (\operatorname{sgn} \pi)(g_{\pi(1)}, \cdots, g_{\pi(m+n)}),$$

where the sum is over all (m, n)-shuffles π. This extends uniquely to a bilinear map $\beta G^r \times \beta G^r \to \beta G^r$. Note the μ is an associative multiplication and satisfies $\partial\mu(c^m, c^n) = \mu(\partial c^m, c^n) + (-1)^m \mu(c^n, \partial c^n)$ if $I(c^m) \cap I(c^n) = \varnothing$. We write $c^m \cdot c^n = \mu(c^m, c^n)$.

By a *p-string* we will mean a p-tuple $(c_1, \cdots, c_p)$ of positive cells, such that $I(c_1) < \cdots < I(c_p)$. A string will be said to be *degenerate* if at least one of its cells is degenerate. We make the convention that there is a unique 0-string, written ( ), and it is nondegenerate. We let $\beta(\beta G^r)$ be the double complex defined by letting $\beta(\beta G^r)_{p,q}$ be the free abelian group generated by all p-strings $(c_1, \cdots, c_p)$ with $\deg c_1 + \cdots + \deg c_p = q$ (where $\deg c = n$ if c is an n-cell), modulo the subgroup generated by degenerate strings, and by letting the boundary operators

$$\partial' : \beta(\beta G^r)_{p,q} \to \beta(\beta G^r)_{p-1,q} \quad \text{and} \quad \partial'' : \beta(\beta G^r)_{p,q} \to \beta(\beta G^r)_{p,q-1}$$

be

$$\partial'(c_1, \cdots, c_p) = \sum_{i=1}^{p-1} (-1)^{i + \deg c_1 + \cdots + \deg c_i} (c_1, \cdots, c_i c_{i+1}, \cdots, c_p),$$

$$\partial''(c_1, \cdots, c_p) = \sum_{i=1}^{p} (-1)^{i-1+\deg c_1 + \cdots + \deg c_{i-1}} (c_1, \cdots, \partial c_i, \cdots, c_p).$$

The following is the main result:

THEOREM 1. $H_i(F\Gamma_1^r) \simeq H_i(\beta(\beta G^r))$, $i \geq 0$.

The homology of the double complex $\beta(\beta G^r)$ means (as usual) the homology of the associated simple complex defined by

$$\beta(\beta G^r)_k = \bigoplus_{p+q=k} \beta(\beta G^r)_{p,q} \quad \text{and} \quad \partial = \partial' + \partial''.$$

2. Two spectral sequences. Now we describe two spectral sequences relating the homology of βG^r and the homology of $\beta(\beta G^r)$ which are analogous to known spectral sequences relating the homology of a DGA algebra and the homology of its bar construction.

The first comes from considering $\beta(\beta G^r)$ as a simple complex with filtration $F_p(\beta(\beta G^r)) = \sum_{p' \leq p} \beta(\beta G^r)_{p',q}$. Let $\beta(\beta G^r)_{p,*}$ denote the complex whose qth chain group is $\beta(\beta G^r)_{p,q}$ and whose boundary is ∂''. Then the spectral sequence associated with this filtered complex satisfies $E^1_{p,q} = H_q(\beta(\beta G^r)_{p,*})$ and $E^\infty =$ the bigraded group associated to a filtration of $H(\beta(\beta G^r))$.

Let Y denote the complex defined by $Y_k = (\beta G^r)_{k-1}$, $k \geqq 2$, $Y_k = 0$, $k \leqq 1$ and whose boundary is the boundary of βG^r. Let X_p be the complex defined by $(X_p)_k = (Y \otimes \cdots \otimes Y)_{k+p}$, where the tensor product is taken p times. Then $\beta(\beta G^r)_{p,*}$ is naturally a subcomplex of X_p.

LEMMA. *The inclusion $\beta(\beta G^r)_{p,*} \hookrightarrow X_p$ induces isomorphisms in homology.*

The homology of X_p can be computed from the homology of G^r by the Künneth formula. Hence, combining the above lemma and the main theorem, we get:

THEOREM 2. *There exists a spectral sequence whose E^1 term can be computed in terms of the homology of G^r by the Künneth formula and which converges to the homology of $F\Gamma^r_1$.*

COROLLARY 1. *$F\Gamma^r_1$ is contractible if and only if $H_i(G^r) = 0$, $i > 0$.*

COROLLARY 2. *$F\Gamma^0_1$ is contractible.*

PROOF. By Corollary 1 and [3].

COROLLARY 3. *$H_2(F\Gamma^r_1) \simeq H_1(G^r)$ and there exists an exact sequence*

$$2(H_2(G^r) \otimes H_1(G^r)) \oplus \operatorname{Tor}(H_1(G^r), H_1(G^r)) \to H_3(G^r) \to H_4(F\Gamma^r_1)$$
$$\to H_1(G^r) \otimes H_1(G^r) \to H_2(G^r) \to H_3(F\Gamma^r_1) \to 0.$$

COROLLARY 4. *Let U be an open interval in the circle S^1. Let G^r_S denote the group of C^r orientation preserving diffeomorphisms of S^1, and let*

G_U^r denote the subgroup of diffeomorphisms having support in U. Then the inclusion map induces an isomorphism $\iota: H_1(G_U^r) \to H_1(G_S^r)$.

PROOF. It is easily seen that ι is onto. Given an element g of G_S^r one can construct a foliation on T^2 by identifying the ends of a trivially foliated cylinder *via* g. The obstruction to homotoping this foliation to zero in $H^2(T^2, \pi_2(F\Gamma_1^r)) \simeq H_1(G_U^r)$ defines a left inverse to ι. Q.E.D.

COROLLARY 5. *$H_1(G^\infty)=0$ and the map $H_1(G^r) \to H_1(G^{r-4})$ is the zero map, $r \geqq 4$.*

PROOF. Results of Moser [4] imply these are true for G_S^r in place of G^r. Hence by Corollary 4 they hold for G^r. Q.E.D.

The second spectral sequence comes from considering a filtered complex C. We let C_k be the subgroup of $\bigoplus_{p+q=k} (\beta G^r)_p \otimes \beta(\beta G^r)_q$ generated by all $c_0 \otimes (c_1, \cdots, c_l)$, where c_0 is a p-cell and $(c_1, \cdots, c_l)$ is a string such that $\deg c_0 + \cdots + \deg c_l + l = q$, and where $I(c_0) < I(c_1)$. We define the boundary $\partial: C_k \to C_{k-1}$ by

$$\begin{aligned} \partial(c_0 \otimes (c_1, \cdots, c_l)) &= \partial c_0 \otimes (c_1, \cdots, c_l) \\ &\quad + (-1)^{\deg c_0 + 1} c_0 \otimes \partial(c_1, \cdots, c_l) \\ &\quad + (-1)^{\deg c_0 + 1} c_0 c_1 \otimes (c_2, \cdots, c_l), \\ \partial(c_0 \otimes (\,)) &= \partial c_0 \otimes (\,). \end{aligned}$$

The homology of C is given by $H_0(C) = \mathbf{Z}$ and $H_i(C) = 0$ for $i > 0$. For, $s: C \to C$ is a homotopy operator, where $s(c_0 \otimes (c_1, \cdots, c_l)) = 1 \otimes (c_0, c_1, \cdots, c_l)$ if c_0 is a positive cell and $s(1 \otimes (c_1, \cdots, c_l)) = 0$ (where 1 is the unique 0-cell). We filter C by letting

$$FC_k = C_k \cap \bigoplus_{p+q=k;\, q \leq l} (\beta G^r)_p \otimes \beta(\beta G^r)_q.$$

THEOREM 3. *The spectral sequence associated with this filtered complex satisfies $E^2_{p,q} \approx H_p(\beta(\beta G^r), H_q(\beta G^r))$ and $E^\infty_{p,q} = 0$ except for $E^\infty_{0,0} = \mathbf{Z}$.*

In view of the main theorem this gives:

COROLLARY. *There exists a spectral sequence with $E^2_{p,q} \simeq H_p(F\Gamma_1^r, H_q(G^r))$ and $E^\infty_{p,q} = 0$ except for $E^\infty_{0,0} = \mathbf{Z}$.*

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