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MINIMALITY IN FAMILIES OF SOLUTIONS OF $\Delta u = Pu$ ON RIEMANNIAN MANIFOLDS

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The notion of minimality of solutions of $\Delta u = Pu$ was first introduced by C. Constantinescu and A. Cornea in 1958. On a Riemann surface, M. Nakai has given a complete characterization of the minimality in the monotone closure of the family of all Dirichlet-finite harmonic functions. His characterization is in terms of a positive bounded regular Borel (representing) measure on the Royden boundary of the Riemann surface (cf. [6]). In this paper we announce that not only his work for the harmonic functions can be generalized for the solutions of the elliptic differential equation $\Delta u = Pu$ on a Riemannian manifold, but more significantly, the property of the existence of a minimal function is to a large extent an intrinsic part of the manifold, perhaps a quasi-conformal or quasi-isometric invariant.

Consider a Riemannian manifold R and the elliptic differential equation $\Delta u = Pu$ on R, where P is nonnegative C^1. For simplicity, solutions of $\Delta u = Pu$ will be called solutions. Let R^* be the Royden compactification, $\Gamma = R^* \backslash R$ the Royden boundary and Δ the harmonic boundary of R (cf. [6]). The open subset $\Delta^P = \{q \in \Delta : q$ has a neighborhood U in R^* with $\int_{U \cap R} P < \infty\}$ of Δ introduced in [2] is crucial for solutions.

Following the pattern that Nakai has established in [6] for harmonic functions and using the results of Glasner-Katz [1, Theorems 1 2], we can construct a positive bounded regular Borel (representing) measure m^P on Γ centered at $z_0 \in R$ with support equal to the closure of Δ^P characterized by $u(z_0) = \int_\Gamma u \, dm^P$ for every solution u with finite energy integral (the so-called PE-function). Moreover, using Harnack's inequality we can also construct a nonnegative kernel $K^P(z, q)$ on $R \times \Gamma$ with the property $u(z) = \int_\Gamma K^P(z, q) u(q) \, dm^P(q)$ for all $z \in R$ and for all PE-functions u. Note that when $P \equiv 0$, m^0 and $K^0(z, q)$ are the corresponding measure and kernel for harmonic functions constructed by Nakai in [6].

Nakai's characterization for $HD^\sim$-functions and $HD^\sim$-minimal

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functions can also be generalized for the corresponding $PE^{\check{}}$-functions and $PE^{\check{}}$-minimal functions.

DEFINITION. A nonnegative solution is called a $PE^{\check{}}$-function if it is the infimum of a downward directed family of PE-functions. The collection of all $PE^{\check{}}$-functions on R is denoted by $PE^{\check{}}(R)$.

THEOREM. *If χ_C is the characteristic function of a compact subset C of Δ^P, then*

$$u(z) = \int_\Gamma K^P(z, q)\chi_C(q)\, dm^P(q)$$

is a $PE^{\check{}}$-function.

THEOREM. *If $u \in PE^{\check{}}(R)$, then*

$$u(z) = \int_\Gamma K^P(z, q)\left(\limsup_{x\in R;\, x\to q} u(x)\right) dm^P(q)$$

for all $z \in R$.

DEFINITION. A nonzero $PE^{\check{}}$-function u is called a $PE^{\check{}}$-minimal function if for any $PE^{\check{}}$-function v such that $u \geqq v$ we have $cu = v$ for some constant c.

THEOREM. *There exists a $PE^{\check{}}$-minimal function on R if and only if there exists a point in Δ^P with positive m^P-measure.*

More precisely, if u is $PE^{\check{}}$-minimal, then there is a point $q_0 \in \Delta^P$ with $m^P(q_0) > 0$ and $u(z) = aK^P(z, q_0)$ for some constant a. Conversely, if $m^P(q_0) > 0$ for some $q_0 \in \Delta^P$, then $K^P(z, q_0)$ is a $PE^{\check{}}$-minimal function.

THEOREM (*Intrinsic property of minimal functions*). *For any $q_0 \in \Delta^P$, $m^P(q_0) > 0$ if and only if $m^0(q_0) > 0$.*

This theorem is a special case of the following stronger result.

THEOREM. *For any connected set $S \subset \Delta^P$, $m^P(S) > 0$ if and only if $m^0(S) > 0$.*

OUTLINE OF THE PROOF. Since the representing measures are regular, we may assume without loss of generality that S is compact.

To prove the necessity, observe that nonnegative harmonic functions, being supersolutions, dominate solutions with the same values on Δ. It can thus be proved that $m^0(S) \geqq m^P(S)$.

The proof of the sufficiency is divided into four steps.

Step 1. When $\int_R P < \infty$, Royden has proved in [5] (also cf. Nakai [3]) that $HB(R)$ and $PB(R)$ are isometrically isomorphic with respect to the sup norm. Under this isometric isomorphism, it can be shown that those HBD-functions and PBE-functions which have the same values on Δ correspond to one another. Consequently, those $HD^{\sim}$- and $PE^{\sim}$-functions which are characterized by the same compact subsets of Δ^P correspond to one another. In other words, a compact subset of Δ^P is of positive m^0-measure if and only if it is of positive m^P-measure.

Step 2. It is now a matter of reducing the problem to the case of $\int_R P < \infty$. A modification of Nakai's result [4, Proposition 9], shows that if $m^0(S) > 0$, where $S \subset \Delta$ is connected and compact, then for any open set U in R^* containing S, there is an open set V in R^* containing S such that $V \subset U$ and $V \cap R$ is a region. In particular, we can find such an open set V that $\int_{V \cap R} P < \infty$ because S is a compact subset of Δ^P.

Step 3. Let $G = V \cap R$. G, being itself a Riemannian manifold, have its own Royden compactification G^*, Royden boundary Γ_G and representing measures m_G^0, m_G^P on Γ_G. Nakai [4, Propositions 7, 8] has observed that there is a unique continuous mapping j of G^* onto cl G, the closure of G in R^*, which fixes G elementwise. Moreover, j is a homeomorphism from $G \cup j^{-1}((\text{cl } G \backslash \text{cl } \partial G) \cap \Gamma)$ to $G \cup ((\text{cl} G \backslash \text{cl } \partial G) \cap \Gamma)$. A nontrivial generalization of Nakai's work [4, Proposition 8] implies that S has positive m^P-measure if and only if $j^{-1}(S)$ has positive m_G^P-measure. In particular, $m_G^0(j^{-1}(S)) > 0$ since $m^0(S) > 0$.

Step 4. By Step 1, $m_G^0(j^{-1}(S)) > 0$ implies that $m_G^P(j^{-1}(S)) > 0$. This in turn implies that $m^P(S) > 0$ by Step 3.

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