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ON A CLASSIFICATION OF A BAIRE SET OF DIFFEOMORPHISMS

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Let $\mathrm{Diff}^r(M^n)$ be the space of C^r diffeomorphisms of a compact C^∞ n-manifold M with the C^r topology, $1 \leqq r \leqq \infty$. Central problems in the study of differentiable dynamical systems, as formulated by Smale [8], [9] are:

(a) Find a Baire subset B of $\mathrm{Diff}^r(M^n)$ with strong stability properties.

(b) Find a practical means of classifying the elements of B.

Smale's survey article [8] is a general reference to these problems and to the definitions of the basic notions used in this abstract.

During the last decade, a number of unsuccessful candidates for B have been studied. Among these are Morse-Smale diffeomorphisms, structurally stable diffeomorphisms, maps satisfying "Axiom A", and Ω-stable diffeomorphisms. The latter two classes were shown to be nongeneric in $\mathrm{Diff}^r(M^n)$ by Abraham and Smale [1], for $r \geqq 1$, $n \geqq 4$, and by Newhouse [7], for $r \geqq 2$, $n = 2$. However, it has been emphasized ([9], for example) that many more counterexamples to the genericity of Ω-stable and Axiom A diffeomorphisms must be constructed for the theory to advance, especially since each new conjecture for B has arisen from careful analysis of past counterexamples. The examples described below are the first such counterexamples in $\mathrm{Diff}^1(M^3)$. More significantly, all the above classes of diffeomorphisms conjectured to solve problem (a) have had the following property: All maps close enough to any diffeomorphism in the class have the same number of periodic points of each period as the original map. Theorem 1 illustrates that this is not a generic property.

THEOREM 1. *Let* $1 \leqq r \leqq \infty$. *For* $f \in \mathrm{Diff}^r(T^3)$ *and positive integer* n, *let* $N_n(f) =$ *number of fixed points of* $f^n = f \circ f \circ \cdots$ *(n-times)* $\cdots \circ f$: $T^3 \to T^3$. *Then, there exists an open set* U *in* $\mathrm{Diff}^r(T^3)$ *such that if* $f_1 \in U$ *and* U_1 *is any neighborhood of* f_1 *in* U, *there is* $f_2 \in U_1$ *and integer* n *such that* $N_n(f_1) \neq N_n(f_2)$ *and all periodic points of* f_2 *of period* $\leqq n$ *are hyperbolic.*

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Before discussing the proof of Theorem 1, let us see what effect it has on problem (b), the classification problem. In [8], Smale conjectured that an effective means of classifying the maps in B might be the zeta function. The *zeta function* of a diffeomorphism f, $\zeta(t) = \zeta(f)(t) = \exp(\sum_{i=1}^{\infty} N_i t^i / i)$ where $N_i = N_i(f)$ as in Theorem 1. Artin and Mazur [2] have shown that a dense set of diffeomorphisms have zeta functions with a positive radius of convergence. Guckenheimer [4] has shown that if f satisfies Axiom A and the no-cycle property, $\zeta(f)$ is rational. However, in order to be at all effective and practical as a means of classification, $\zeta(f)$ must be rational for a Baire set of diffeomorphisms. Whether or not, $\zeta(f)$ is generally rational was asked in [8, Problem 4.5], [10], [11], and [12]. Theorem 2 uses Theorem 1 to answer this question.

THEOREM 2. *Diffeomorphisms with rational zeta functions do not form a Baire subset of* $\mathrm{Diff}^r(T^3)$, $1 \leqq r \leqq \infty$.

PROOF OF THEOREM 2. Enumerate the countable set [3] of rational zeta functions as $Z_1, Z_2, \cdots$, where $Z_j(t) = \exp(\sum_{i=1}^{\infty} N_i^j t^i / i)$. Let U be the open set in $\mathrm{Diff}^r(T^3)$ from Theorem 1. Let $V_j = \{f \in U \mid \zeta(f) \neq Z_j$ and there is an i such that f^i has only hyperbolic fixed points with $N_i(f) \neq N_i^j\}$. By the hyperbolicity in the definition of V_j, each V_j is open. Using the Kupka-Smale Theorem and Theorem 1, it is easily seen that each V_j is also dense. Then, $V = \bigcap V_j$ is a Baire subset of U; and no diffeomorphism in V can have a rational zeta function.

Finally, Theorem 3 below deals with another aspect of the classification problem. It states that Ω-conjugacy is not a reasonable equivalence relation to use in classifying diffeomorphisms. The same result holds for any equivalence relation which has all $N_n(f)$ constant in each equivalence class. The proof of Theorem 3 is the same as that of Theorem 2 with N_i^j replaced by $N_i(h_j)$.

THEOREM 3. *There does not exist a countable set* $\{h_j\}$ *and a Baire subset* B *in* $\mathrm{Diff}^r(T^3)$ *such that each* f *in* B *is* Ω*-conjugate to some* h_j.

SKETCH OF PROOF OF THEOREM 1. Let $A: T^2 \to T^2$ be a hyperbolic toral automorphism, i.e., a map on T^2 induced by a matrix in $\mathrm{Gl}(2, Z)$ with determinant 1 and eigenvalues off the unit circle. The stable manifolds $\{W^s\}$ of A give a foliation [5] of T^2. Let $g: T^2 \to T^2$ be the corresponding "derived from Anosov" map [13], [8]. g respects the above foliation of A, i.e., maps leaves to leaves, but has two new fixed points, x_0 and y_0. The nonwandering set of g, $\Omega(g)$, contains a one-dimensional attractor Σ. Let $h: S^1 \to S^1$ be a hyperbolic diffeomorphism with one sink, $\{-1\}$, and one source, $\{+1\}$. Then, $g \times h$ is a hyperbolic diffeomorphism of $T^3 \cong T^2 \times S^1$. Let b be a C^∞ bump function on T^3 which is the identity near $\Sigma \times \{+1\}$ and near $T^2 \times \{-1\}$, but which forces the two-dimensional local unstable manifold of $(x_0, +1)$ to intersect the one-dimensional stable manifold of $(x_0, +1)$ transversally. One constructs b so that it preserves the foliation of $T^2 \times S^1$ whose leaves are $W^s \times S^1$. Let $f = b \circ (g \times h)$. Diagram 1 shows the local stable and unstable manifold structure for f around $(x_0, +1)$; and Diagram 2 pictures this structure on a piece of the leaf F of the foliation that contains $(x_0, 1)$. $f: \Omega(f|F) \to \Omega(f|F)$ is topologically conjugate to the shift map on 3^Z. The "circle" c in Diagram 1 lies in

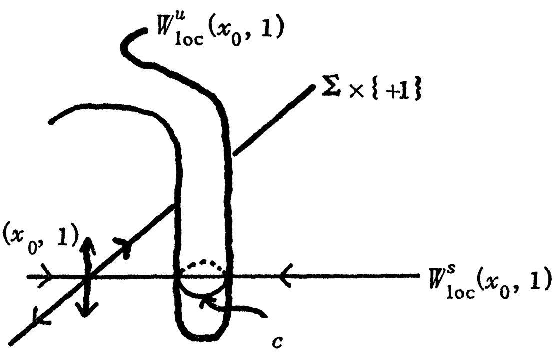

DIAGRAM 1

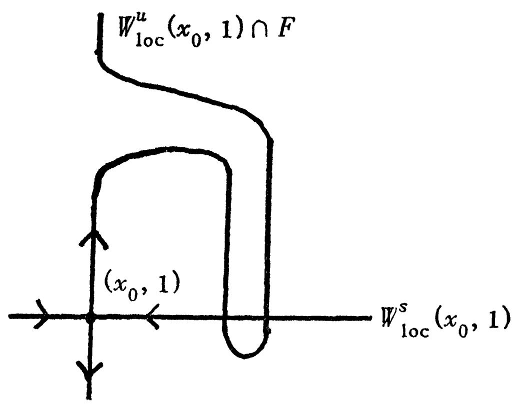

DIAGRAM 2

$W^u_{\text{loc}}(x_0, 1) \cap T^2 \times \{+1\}$ and intersects a (one-dimensional) $W^s_{\text{loc}}(y)$ nontransversally for some $y \in \Sigma \times \{+1\}$. This point of intersection is nonwandering by the "cloud lemma" [1], [8] since $W^s(x_0, 1)$ and

$W^u(y)$ meet transversally on $\Sigma \times \{+1\}$. Because the intersection was not transversal, this nonwandering point is not hyperbolic; and therefore f does not satisfy Axiom A.

f is normally-hyperbolic with respect to the foliation [6]. Let U be a neighborhood of f in $\mathrm{Diff}^r(T^3)$ such that for f' in U the stable and unstable manifolds of $(x_0, 1)$ meet as above and f' respects a foliation near the foliation of f. By similar arguments, no f' in U satisfies Axiom A. If $f_1 \in U$, one chooses f_2 as close as one wishes to f_1 and equal to f_1 near the one-dimensional attractor "$\Sigma \times \{+1\}$" of f_1. Let y_1 correspond for f_1 and y_2 correspond for f_2 to the above y for f. Choose f_2 so that y_1 and y_2 do not lie on the same stable manifold. Then, there exists a point z, periodic for both f_1 and f_2, whose local stable manifold lies between that of y_1 and that of y_2. If n is the period of z, F_1 is the leaf of z for the foliation of f_1, and F_2 is the leaf of z for the foliation of f_2, one checks that $f_1^n | F_1: F_1 \to F_1$ has a different number of fixed points than $f_2^n | F_2: F_2 \to F_2$. The construction of f_2 relies on stable manifold theory and transversality. In perturbing f_1 to f_2, one can be careful and keep track of periodic points of period n off the leaf F_1 so that $N_n(f_1)$ will differ from $N_n(f_2)$. Finally, one achieves the hyperbolicity mentioned in Theorem 1 by using the Kupka-Smale Theorem in the construction of f_2.

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