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TIGHT EQUIVARIANT IMMERSIONS OF SYMMETRIC SPACES

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Introduction. Let G/K be a compact, irreducible symmetric space and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the Lie algebra of G. If π is a nontrivial real class-one representation of G on E^N with $0\neq e$, K-fixed, then the map $\pi:G/K \to E^n$ given by $gK\to\pi(g)e$ gives an immersion of G/K into E^N. The purpose of this note is to announce the classification of such immersions with minimal absolute curvature (i.e., are tight) [1], [4].

In a slightly different vein is the problem of finding to what symmetric spaces can the work of Frankel [2] be extended. One can describe Frankel's method as "take an equivariant immersion of a homogeneous space and examine the critical manifolds for nondegenerate height functions." The present work shows that to extend Frankel's results to spaces which are not R-spaces, the exceptional groups for instance, will require some modification of method.

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Tightness. If M is a compact, n-dimensional connected C^∞ manifold and ϕ is a real C^∞ function on M with nondegenerate critical points then

$$\beta_k(\phi) = \text{number of critical points of index } k \text{ of } \phi,$$

$$\beta_k(M) = \text{minimum } \{\beta_k(\phi) \mid \phi \text{ nondegenerate}\},$$

$$\beta(M) = \text{minimum } \left\{ \sum_{k=1}^{n} \beta_k(\phi) \mid \phi \text{ nondegenerate} \right\}.$$

If $f:M\to R^N$ is an immersion the height functions on M are the functions

$$\phi_a(x) = (a, f(x)) \quad \text{where } a \in R^N.$$

f is tight (k-tight) if $\beta(\phi_a)=\beta(M)$ $(\beta_k(\phi_a)=\beta_k(M))$ when ϕ_a is nondegenerate.

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Symmetric R-spaces. Let $\mathfrak{L}$ be a real noncompact semisimple Lie algebra with $Z\in\mathfrak{L}$ such that ad Z is semisimple with eigenvalues 0, ± 1. There is a Cartan decomposition $\mathfrak{L}=\mathfrak{g}+\mathfrak{F}$ with $Z\in\mathfrak{F}$. If L is a Lie group without center and with Lie algebra $\mathfrak{L}$ and G is the subgroup of L corresponding to $\mathfrak{g}$ and if $K=\{g\in G \mid \text{Ad } gZ=Z\}$ then G/K is symmetric, and is called a symmetric R-space. We have of course the imbedding

(A) $$G/K \to \mathfrak{F} \quad \text{by} \quad gK \to \text{Ad } gZ.$$

Second fundamental form. If $f: M\to N$ is an immersion of a manifold in a Riemannian manifold and $N_x = M_x \oplus M_x^\perp$ is the decomposition of the tangent space at $x\in N$ under the Riemannian metric then it is convenient to regard the second fundamental form of the immersion at x as the bilinear symmetric map $\alpha: M_x\times M_x\to M_x^\perp$ constructed as follows: if X, $Y\in M_x$, and $\tilde{X}$, $\tilde{Y}$ are tangential vector fields with $\tilde{X}_x=\tilde{X}$, $\tilde{Y}_x=Y$, then

$\alpha(X, Y)$ = normal component of $(\overline{\nabla}_X Y)_x$ where $\overline{\nabla}$ is the Riemannian connection on N.

THEOREM 1. *Let G/K be an irreducible compact symmetric space and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the Lie algebra of G. If π is a real class-one representation of G on E^N with $0\neq e$, K-fixed, and π also denotes the corresponding representation of g, then for the immersion $\pi: G/K\to E^N$ given by $gK\to\pi(g)e$*

$$\alpha(X, Y) = \pi(X)\pi(Y)e \quad \textit{for} X,\ Y \in \mathfrak{p}.$$

The following theorem is an useful improvement of Theorem 4 [4].

THEOREM 2. *If $f: M\to R^N$ is a 0-tight immersion of a compact connected manifold then there is an open set $U\subset M$ such that the second fundamental form is an onto map at each point of U.*

REMARK ON PROOF OF THEOREM 2. This theorem represents a change in the point of view from [4] more than anything else. Using local coordinates $\{x_i\}$ about the point m in M

$$\alpha\left(\left(\frac{\partial}{\partial x_i}\right)_m, \left(\frac{\partial}{\partial x_j}\right)_m\right) = \text{normal component of } \frac{\partial^2 f}{\partial x_i \partial x_j} \text{ at } m.$$

Now if ϕ_a is a nondegenerate height function with its maximum at m then its Hessian at m is the matrix

$$\left[\left(a, \alpha\left(\left(\frac{\partial}{\partial x_i}\right)_m, \left(\frac{\partial}{\partial x_j}\right)_m\right)\right)\right].$$

If α is not onto we can find $z \in M_m$ such that ϕ_{a+z} has a nondegenerate critical point of index n at m but its absolute maximum elsewhere. Then by standard "Morse Theory techniques" we can get a ϕ_v close to ϕ_{a+z} such that ϕ_v is nondegenerate.

Main result. One can reduce the classification of tight equivariant immersions of irreducible symmetric spaces to the examination of irreducible representations and we get

THEOREM 3. *Let G/K be an irreducible compact locally symmetric space and π an irreducible class-one representation of G giving the immersion $\pi: G/K \to E^N$. Then the following are equivalent:*

(i) *π is 0-tight.*

(ii) *G/K is a symmetric R-space and π is one of the imbeddings* (A).

(iii) *π is tight (has minimal total curvature).*

REMARK ON PROOF. (ii)$\Rightarrow$(iii) was proved in [3].

(iii)$\Rightarrow$(i). See [4] and [5].

(i)$\Rightarrow$(ii). The central idea in the proof is the following: Let $\mathfrak{L} = \mathfrak{g} \oplus E^N$. Give $\mathfrak{L}$ the following algebraic structure:

(i) X, Y in $\mathfrak{g}$: $[X, Y]$ as in $\mathfrak{g}$.

(ii) X in $\mathfrak{g}$, u in E^N: $[X, u] = -[u, X] = \pi(X)u$.

(iii) u, v in E^N then $[u, v]$ is in $\mathfrak{g}$ where

$$-B([u, v], X) = (v, \pi(X)u) \quad \text{for all } X \text{ in } \mathfrak{g},$$

where the inner product on the right is the Euclidean inner product on E^N and B is the Killing-form on $\mathfrak{g}$. Then we have

LEMMA 1. *Under the assumptions of Theorem* 3, *Part* (i), *this algebraic structure makes $\mathfrak{L} = \mathfrak{g} + E^N$ into a Lie algebra.*

Details will appear elsewhere.

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