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K-THEORY OF A SPACE WITH COEFFICIENTS IN A (DISCRETE) RING

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In [2], [3], S. Gersten has introduced higher K-groups of a ring which satisfy properties analogous to those of a generalized homology theory in a suitably defined homotopy category of rings [1]. In this announcement we use Gersten's K-groups to define for a ring R a generalized cohomology theory $K_R^*(\ )$, analogous to the Atiyah-Hirzebruch K-theory, on the category of finite simplicial sets so that $K_R^*(pt)=K_\Sigma^*R$, where K_Σ^*R are Gersten's stable K-groups of the ring R. If R is suitably restricted, in particular if it is commutative and regular, the theory $K_R^*(\ )$ will have products and Adams operations. One may also define, using the continuous theory in [6], a K-theory $K_\Lambda^*(\ )$ with coefficients in a Banach ring Λ. This theory coincides with the Atiyah-Hirzebruch theory for $\Lambda=\boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$. We give here an outline of proofs. A full account will appear elsewhere.

1. **Definition of the theory.** We recall the definition of Gersten's theory as given in [5]. Let R be a ring (without unit). The functor $R\mapsto R[t]$ together with the natural transformations $R[t]\to R$ via "$t\to 1$", and $R[t]\to R[t, t']$ via $t\to tt'$ define a cotriple in the category of rings. If ER is the ideal $R[t]t$, then the restriction of those maps makes the functor $R\mapsto ER$ a cotriple. Associated to these cotriples are canonical simplicial rings $R[T]$ and $\overline{E}R$ with

$$R[T]_n = R[t_0, \cdots, t_n], \qquad \overline{E}R_n = E^{n+1}R.$$

Let QR be the simplicial ring

$$QR = R[T]/\overline{E}R.$$

One has

$$K^{-i-1}R = \pi_i \ \mathrm{Gl}\ QR$$

where Gl denotes the general linear group functor. This K-theory of rings is stabilized as follows [3]. Let ΓR be the kernel of $R[t, t^{-1}]\to R$. Then there is a natural homomorphism

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$$K^iR \xrightarrow{\beta} K^{i-1}\Gamma R, \qquad i \leqq 0,$$

analogous to the Bott map, which is an isomorphism when R is K-regular.

Put

$$K_\Sigma^i R = \operatorname{inj}\lim_n K^{i-n}\Gamma^n R, \qquad -\infty < i < \infty.$$

Then if $R \to S \to T$ is a Gl-fibration [2], there is a long exact sequence

$$\cdots \to K_\Sigma^i T \to K_\Sigma^i S \to K_\Sigma^i R \xrightarrow{\partial} K_\Sigma^{i+1} \to \cdots.$$

Now to define a cohomology theory for simplicial sets, we will give a contravariant functor (; R) from simplicial sets to rings which

(1) sends coproducts to products,
(2) sends cofibrations to Gl-fibrations,
(3) sends a point to R.

For X and Y simplicial sets, let $\Delta(X; Y)$ denote the set of all simplicial maps from X to Y. Put

$$(X; R) = \Delta(X; QR).$$

Then $(X; R)$ is a ring and $(pt; R) = R$, since $QR_0 = R$.

DEFINITION 1.1. For X a finite simplicial set, $K_R^*(X) = K_\Sigma^*(X; R)$. The long exact sequence of a cofibration arises from

PROPOSITION 1.2. *If $Y \to X \to X/Y$ is a cofibration of simplicial sets then*

$$(X/Y; R) \to (X; R) \to (Y; R)$$

is a Gl-*fibration.*

This proposition follows from the following properties of the functor $\Delta($; $)$.

LEMMA 1.3. *If F is a functor which is left exact and preserves products then*

$$\Delta(X; FY) = F\Delta(X; Y).$$

PROOF. Follows from the fact that F preserves equalizers.

LEMMA 1.4. *If Y is a simplicial object in a category with a forgetful functor to sets and Y is contractible as a set complex then $\Delta($;$Y)$ is an exact functor.*

To verify the homotopy axiom for the theory we must prove

PROPOSITION 1.5. *If $X \rightarrow Y$ is a map of finite simplicial sets which induces an isomorphism $H_*(X; \mathbf{Z}) \rightarrow H_*(Y; \mathbf{Z})$, then $f^! : K_R^*(Y) \rightarrow K_R^*(X)$ is an isomorphism.*

This proposition follows immediately from the fact that we have an analogue of the Atiyah-Hirzebruch spectral sequence defined intrinsically in the theory K_R^* as follows.

Let X^n be the n-skeleton of X. We have a tower of Gl-fibrations

$$\cdots \longrightarrow (X^n; R) \longrightarrow (X^{n-1}; R) \longrightarrow \cdots \longrightarrow (X^0; R).$$

The long exact K_Σ^*-theory sequences of these fibrations define a homology exact couple. The spectral sequence of that couple converges strongly to $K_R^*(X)$. One has $E_1^{p,q} = K_\Sigma^{p+q}(X^p/X^{p-1}; R)$. By a brute force calculation

LEMMA 1.6. *$K_\Sigma^{p+q}(X^p/X^{p-1}; R) = \bigoplus_\sigma K_\Sigma^{p+q}R$, where the sum runs over all nondegenerate p-simplexes σ of X.*

Standard diagram chases now establish

THEOREM 1.7. *There is a natural spectral sequence $\{E_r\}$ converging to $K_R^*(X)$ with*

$$E_2^{p,q} = H^p(X; K_\Sigma^q R).$$

Thus

THEOREM 1.8. *$K_R^*(\ )$ is a generalized cohomology theory on the category of finite simplicial sets.*

In addition,

THEOREM 1.9. *$K_R^*(\ )$ depends only on the ring homotopy type of R and if $R \rightarrow S \rightarrow T$ is a* Gl-*fibration of rings there is a natural exact triangle of theories*

$$\begin{array}{ccc} K_R^*(\ ) & \longrightarrow & K_S^*(\ ) \\ & {}^{\delta}\nwarrow \qquad \swarrow & \\ & K_T^*(\ ) & \end{array}$$

where δ has degree $+1$.

2. **Products and Adams operations.** Let R and T be rings, X and Y simplicial sets. We then have a pairing

$$(X; R) \otimes_Z (Y; T) \xrightarrow{\phi} (X \times Y; R \otimes T)$$

given by

$$\phi(\alpha \otimes \beta)(x, y) = \alpha(x) \otimes \beta(y).$$

Using the product structure in $K_\Sigma^*(\ )$ [**4**] one has

THEOREM 2.1. *There is a natural graded associative pairing*

$$K_R^*(X) \otimes K_T^*(Y) \to K_{R\otimes T}^*(X \times Y).$$

If R is a commutative ring there is a natural graded commutative ring structure on $K_R^*(X)$ arising from the diagonal $\Delta: X \to X \times X$.

Now suppose that R is a K-regular ring [**2**]. From a truncated version of the spectral sequence of Theorem 1.7 one has

THEOREM 2.2. *If R is K-regular,*

$$K_\Sigma^i(X; R) = K^i(X; R)$$

for $i \leqq 0$.

Now the theory K^i has Adams operations which are graded ring homomorphisms. Let $K_R^-(\ )$ be the nonpositive graded part of $K_R^*(\ )$. Then

THEOREM 2.3. *If R is K-regular there are natural graded ring morphisms*

$$\psi^k: K_R^-(X) \to K_R^-(X)$$

for $k \geqq 0$. The ψ^k commute with the boundary of the long exact sequence of a cofibration when that makes sense.

REMARK 2.4. Using the continuous polynomials of [**6**] one may define a theory $K_\Lambda^*(\ )$ for Λ a valuation ring. For $\Lambda = \boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ there is a natural equivalence

$$K_\Lambda^*(\ ) \to K\Lambda^*(\ )$$

where $K\Lambda^*$ is the K-theory of Atiyah and Hirzebruch. It would be interesting to know the coefficient group $K_\Lambda^*(pt) = K^*\Lambda$ for $\Lambda = \boldsymbol{Q}_p$ or $\boldsymbol{Z}_p$.

REMARK 2.5. The ring complex QR above may be replaced by the nicer ring complex ΔR where

$$\Delta R_n = R[t_0, \cdots, t_n]/t_0 + \cdots + t_n - 1,$$

and

$$\begin{aligned} d_i t_j &= t_j, & i &> j, \\ &= 0, & i &= j, \\ &= t_{j-1}, & i &< j, \end{aligned}$$

$$\begin{aligned} s_i t_j &= t_j, & i &> j, \\ &= t_j + t_{j+1}, & i &= j, \\ &= t_{j+1}, & i &< j. \end{aligned}$$

One may now redefine $(X; R)$ as the ring of simplicial maps of X to ΔR. The same K-theory for X now arises in view of

PROPOSITION 2.6. $\pi_i \operatorname{Gl} \Delta R = K^{-i-1}R$, $i \geqq 0$.

This proposition is proved by showing that $\pi_i \operatorname{Gl} \Delta R$ satisfies the axioms for $K^{-i-1}R$ [2].

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