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CHEVALLEY GROUPS OVER COMMUTATIVE RINGS[1]

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1. **Introduction.** Steinberg [8] has given a simple presentation for the universal central extension [7], [8], [9] of the group of rational points of a simply connected Chevalley group over a field. In this note we announce a similar theory for the simply connected Chevalley groups over *commutative rings* and outline the proof of a stability theorem for certain functors resulting from this construction. Complete proofs will appear elsewhere.

Let us introduce some notation. A denotes a commutative ring with 1, A^* is its group of invertible elements, $\mathfrak{p}$ and $\mathfrak{q}$ are ideals of A, and $(1+\mathfrak{q})^*=(1+\mathfrak{q})\cap A^*$. Φ is a reduced irreducible root system [2] and $G(\Phi,\ )$ is the simply connected Chevalley-Demazure group scheme with root system Φ. If Φ is of type C_l, $l\geqq 1$ $(C_1=A_1)$, we say Φ is *symplectic*, and if Φ is of type A_l, B_l, C_l, or D_l, we say Φ is *classical*. The subgroup of $G(\Phi, A)$ generated by the elementary unipotents $e_\alpha(t)$, $\alpha\in\Phi$, $t\in A$, will be denoted $E(\Phi, A)$. A full discussion of these notions may be found in [3], [5], and [9].

Define the Steinberg group, $\mathrm{St}(\Phi, A)$, to be the group with generators $x_\alpha(t)$, $\alpha\in\Phi$, $t\in A$, subject to the relations

$$(1.1)\quad \begin{array}{lll} x_\alpha(s)x_\alpha(t) &= x_\alpha(s+t) & (\alpha\in\Phi;\ s,t\in A) \\ {[x_\alpha(s), x_\beta(t)]} &= \prod x_{i\alpha+j\beta}(N_{\alpha,\beta,i,j}s^it^j) & (\alpha,\beta\in\Phi,\ \alpha+\beta\neq 0) \end{array}$$

where the product is as in [8]. Since the elementary unipotents $e_\alpha(t)$ also satisfy these relations, the map $x_\alpha(t)\mapsto e_\alpha(t)$ extends to a homomorphism $\pi:\mathrm{St}(\Phi, A)\to G(\Phi, A)$ with image $E(\Phi, A)$. Set $\ker \pi = L(\Phi, A)$.

In §2 we present certain commutator formulas which yield necessary and sufficient conditions for $E(\Phi, A)$ and $\mathrm{St}(\Phi, A)$ to be their own derived groups. In §3 we show that the extension $\mathrm{St}(\Phi, A)\to E(\Phi, A)$

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is "stably central," and, under certain restrictions on rk Φ, that every central extension of $\mathrm{St}(\Phi, A)$ splits. This generalizes results of Milnor [6], Kervaire [4] and Steinberg [9] for $\mathrm{SL}(n, A)$, $n \geqq 3$ (the Chevalley groups of type A_n, $n \geqq 2$). The functor $\lim_{n\to\infty} L(A_n, \ )$ is Milnor's algebraic K_2 functor, and similar functors arise from the results of §2 and §3 for the other Chevalley groups.

Now suppose Φ_l is a root system of rank l. An inclusion $\Phi_l \subset \Phi_{l+1}$ induces homomorphisms of the corresponding groups and, in particular, a map $\theta_l: L(\Phi_l, A) \to L(\Phi_{l+1}, A)$. When A is a field, Steinberg [8] has shown that the maps θ_l are always surjective. This partially answers a stability question: how large must l be, relative to the dimension of the maximal ideal space of A, for θ_l to be surjective? In §4 we show that $l=1$ suffices for almost all semilocal rings—rings with dim max A equal to 0. Consequences of this result will be discussed in §4.

2. **Commutators in Chevalley groups.** Set

$$\mathrm{St}(\Phi, \mathfrak{q}) = \ker(\mathrm{St}(\Phi, A) \to \mathrm{St}(\Phi, A/\mathfrak{q})).$$

The image of $\mathrm{St}(\Phi, \mathfrak{q})$ under π is denoted $E(\Phi, \mathfrak{q})$, and $L(\Phi, \mathfrak{q}) = \ker \pi \cap \mathrm{St}(\Phi, \mathfrak{q})$. We let $E(\mathfrak{p}, \mathfrak{q})$ be the smallest normal subgroup of $E(\Phi, A)$ containing $\{e_\alpha(p) \mid p \in \mathfrak{p}, \alpha \text{ short}\} \cup \{e_\beta(q) \mid q \in \mathfrak{q}, \beta \text{ long}\}$. If Φ has only one root length, by convention all roots are long.

(2.1) $[E(\mathfrak{q}), E(\mathfrak{p})] \supset E((u-1)\mathfrak{p}, (u^n-1)\mathfrak{p})$ *for all* $u \in (1+\mathfrak{q})^*$, *where* $n=1$ *if* Φ *is nonsymplectic and* $n=2$ *if* Φ *is symplectic.*

(2.2) Let $d[\mathfrak{q}]$ be the ideal generated by $\{q^2-q \mid q \in \mathfrak{q}\}$ and let $s[\mathfrak{q}]$ be the ideal generated by $\{q^2 \mid q \in \mathfrak{q}\}$. It has been shown by Bass and Tate [1] that $d[A] = \bigcap \mathfrak{m}$, where $\mathfrak{m}$ ranges over all maximal ideals of A such that $A/\mathfrak{m} \approx F_2$. Moreover, $d[A]\mathfrak{q} \subset d[\mathfrak{q}] \subset \mathfrak{q}$.

Let F be a subgroup of $G(\Phi, A)$ normalized by $E(\Phi, A)$ and, set $F' = [E(\mathfrak{q}), F]$.

(2.3) THEOREM. *Assume* rk $\Phi \geqq 2$, *and suppose for some* $\gamma \in \Phi$ *and some* $q_0 \in A$ *that* $e_\gamma(q_0) \in F$. *Then* $E(\mathfrak{a}, \mathfrak{b}) \subset F'$, *where* $\mathfrak{a} = \mathfrak{b} = \mathfrak{q}q_0$, *except in the following cases*:

$$\begin{array}{lll}
\gamma \text{ long}: & \Phi = G_2 & \mathfrak{a} = d[\mathfrak{q}]q_0 \\
& \Phi = C_2 & \mathfrak{a} = d[\mathfrak{q}]q_0 + \mathfrak{q}d[Aq_0] \\
& & \mathfrak{b} = d[\mathfrak{q}]q_0^2 + \mathfrak{q}d[Aq_0]q_0 + 2\mathfrak{q}q_0 \\
& \Phi = C_l, \quad l > 2 & \mathfrak{b} = \mathfrak{q}q_0^2 + 2\mathfrak{q}q_0 + s[\mathfrak{q}]q_0 \\
\gamma \text{ short}: & \Phi = G_2 & \mathfrak{a} = 2\mathfrak{q}q_0 + d[\mathfrak{q}]q_0^3 + 3d[\mathfrak{q}]q_0 \\
& & \mathfrak{b} = \mathfrak{q}q_0^3 + 3\mathfrak{q}q_0
\end{array}$$

$$\Phi = C_2 \qquad \mathfrak{a} = d[\mathfrak{q}]q_0 + \mathfrak{q}d[Aq_0]$$
$$\mathfrak{b} = d[\mathfrak{q}]q_0^2 + \mathfrak{q}d[Aq_0]q_0 + 2\mathfrak{q}q_0$$
$$\Phi = B_l, C_l, F_4, \quad l > 2 \qquad \mathfrak{b} = \mathfrak{q}q_0^2 + 2\mathfrak{q}q_0.$$

Moreover, if A has no residue field with two elements, the case $\Phi = C_2$ is the same as $\Phi = C_l$, $l > 2$ when γ is long.

(2.4) COROLLARY. $[E(\Phi, A), E(\Phi, \mathfrak{q})] = E(\Phi, \mathfrak{q})$ *provided, when $\Phi = C_2$ or G_2, that A has no residue field with two elements, and when $\Phi = A_1$, that the elements $u^2 - 1$, $u \in A^*$, generate the unit ideal of A.*

The case of A_1 follows from (2.1); the others from (2.2) and (2.3), with $F = E(A)$, $q_0 = 1$.

REMARKS. (a) For $\mathfrak{q} = A$, the hypotheses of (2.4) are necessary and sufficient. To see this, note that $E(A) \rightarrow E(A/\mathfrak{p})$ is surjective, which shows that $E(A/\mathfrak{p})$ is its own derived group whenever $E(A)$ is. However, it is well known that the groups $\mathrm{SL}(2, \boldsymbol{F}_2)$, $\mathrm{SL}(2, \boldsymbol{F}_3)$, $\mathrm{Sp}(4, \boldsymbol{F}_2)$, and $G_2(\boldsymbol{F}_2)$ contain normal subgroups of index 2.

(b) (2.1), (2.3), and (2.4) remain true if E is replaced by St throughout and F is taken to be a normal subgroup of $\mathrm{St}(\Phi, A)$.

3. **Central extensions and H_2.** Recall [7], [9] that the universal central extension of a group G is a *central extension*, $\hat{G}$, of G which is *its own derived group* and which has *no nonsplit central extensions* of its own. These conditions characterize $\hat{G}$ up to unique isomorphism, and $\ker(\hat{G} \rightarrow G) \approx H_2(G, \boldsymbol{Z})$.

(3.1) THEOREM. *Let Φ be a reduced irreducible root system, of rank $\geqq 5$ if Φ is of type B_l or D_l, and of rank $\geqq 4$ otherwise. Then* $\mathrm{St}(\Phi, A)$ *has no nonsplit central extensions. If A has no residue field with two elements, the same is true for $\Phi = C_3$ or B_4. If the elements $u^2 - 1$, $u \in A^*$, generate the unit ideal of A, then* $\mathrm{St}(\Phi, A)$ *has no nonsplit central extensions whenever* $\mathrm{rk}\, \Phi \geqq 3$.

OUTLINE OF PROOF. Given a central extension $p: F \rightarrow \mathrm{St}(\Phi, A)$, we must construct a section $s: \mathrm{St}(\Phi, A) \rightarrow F$ of p. Over each subgroup of type A_2, B_3, and C_3 we define canonical liftings, $y_\alpha(t)$, of the generators $x_\alpha(t)$ of $\mathrm{St}(\Phi, A)$ belonging to that subgroup and then prove that the liftings so defined are independent of the subgroup chosen. Finally we verify that relations (1.1) hold for the elements $y_\alpha(t)$, showing that $x_\alpha(t) \mapsto y_\alpha(t)$ defines a homomorphism which is the desired section for p. Each step involves technical considerations which differ from root system to root system, accounting for the rather complicated hypotheses of (3.1).

REMARK. For $\mathrm{SL}(n, A)$, (3.1) is due independently to Steinberg [9] and Kervaire [4]. It is possible to weaken the hypotheses on A slightly for the cases $\Phi = A_3, B_3, D_4$ (e.g. $D_4(F_3)$ has no nonsplit central extensions). If A has enough units, the theorem may be extended to groups of rank <3 using the method of [8].

(3.2) THEOREM. *Let Φ_0 be a simple system in Φ* [3], *let $\alpha \in \Phi_0$, and denote by Φ' the subsystem of Φ generated by $\Phi_0 - \{\alpha\}$. Then* $\ker \pi \cap M$ *is central in* $\mathrm{St}(\Phi, A)$, *where M is the image of* $\mathrm{St}(\Phi', A)$ *in* $\mathrm{St}(\Phi, A)$ *under the map induced by $\Phi' \subset \Phi$.*

COROLLARY. *If Φ, A are as in* (3.1), *then* $\mathrm{St}(\Phi, A)/[\ker \pi, \mathrm{St}(\Phi, A)]$ *is the universal central extension of $E(\Phi, A)$. In particular, whenever π is central, $L(\Phi, A) \approx H_2(E(\Phi, A), \boldsymbol{Z})$.*

If Φ is classical, let $\mathrm{St}_\infty(\Phi, A)$, $E_\infty(\Phi, A)$, $L_\infty(\Phi, A)$ be the direct limits as $l \to \infty$ of the groups $\mathrm{St}(\Phi_l, A)$, $E(\Phi_l, A)$, $L(\Phi_l, A)$.

(3.3) COROLLARY. *If Φ is classical, then* $\mathrm{St}_\infty(\Phi, A)$ *is the universal central extension of $E_\infty(\Phi, A)$ and $L_\infty(\Phi, A) \approx H_2(E_\infty(\Phi, A), \boldsymbol{Z})$.*

4. **Stability in dimension 0.** For $u \in A^*$, $\alpha \in \Phi$, define elements $\hat{h}_\alpha(u)$ as in [8]. Let $\hat{H}(\Phi, A)$ be the subgroup of $\mathrm{St}(\Phi, A)$ generated by the $\hat{h}_\alpha(u)$, and let $\hat{H}(\Phi, \mathfrak{q})$ be the smallest *normal* subgroup of $\hat{H}(\Phi, A)$ containing all $\hat{h}_\alpha(v)$, $v \in (1+\mathfrak{q})^*$, $\alpha \in \Phi$.

The pairing $(u, v) \mapsto \{u, v\}_l = \hat{h}_\alpha(uv)\hat{h}_\alpha(u)^{-1}\hat{h}_\alpha(v)^{-1}$ takes values in $L(\Phi_l, \mathfrak{q})$ if u or v is in $(1+\mathfrak{q})^*$ and is independent of the long root α chosen. Denote the subgroup of $L(\Phi_l, \mathfrak{q})$ generated by the values of $\{\ ,\ \}_l$ by $D(\Phi_l, \mathfrak{q})$. $D(\Phi_l, \mathfrak{q})$ is a *central* subgroup of $\mathrm{St}(\Phi_l, A)$ (cf. [9]), and the induced map $D(\Phi_l, \mathfrak{q}) \to D(\Phi_{l+1}, \mathfrak{q})$ is clearly surjective.

If S is a subset of A, we write $\boldsymbol{Z}[S]$ for the subring of A generated by S.

(4.1) THEOREM. *Let $\mathfrak{q}$ be an ideal contained in* rad A. *If Φ is symplectic assume $A = \boldsymbol{Z}[(A^*)^2]$; otherwise assume $A = \boldsymbol{Z}[A^*]$. Then $L(\Phi, \mathfrak{q}) = D(\Phi, \mathfrak{q})$. In particular, the restrictions of the maps θ_l of* §1 *to $L(\Phi_l, \mathfrak{q})$ are surjective.*

(4.2) THEOREM. *Let A be a semilocal ring with at most one residue field isomorphic to $\boldsymbol{F}_2$. If Φ is symplectic, assume further that $A = \boldsymbol{Z}[(A^*)^2]$ (this is automatic if $2 \in A^*$). Then $L(\Phi, A) = D(\Phi, A)$ and the maps θ_l of* §1 *are surjective. Moreover, if Φ and A are as in* (3.1), $\mathrm{St}(\Phi, A)$ *is the universal central extension of $E(\Phi, A)$ and $L(\Phi, A) \approx H_2(E(\Phi, A), \boldsymbol{Z})$.*

Note. Matsumoto [5] has shown the *injectivity* of θ_l when A is a field. A paper of the author's now in preparation describes certain new identities satisfied by $\{\ ,\ \}$ which imply this injective stability theorem for a radical ideal in a semilocal ring generated by its units.

The proof of (4.1) is based on the following decomposition of the group $\mathrm{St}(\Phi, \mathfrak{q})$ when $\mathfrak{q}\subset \operatorname{rad} A$, similar to the Bruhat decomposition [2], [9] of the Chevalley groups.

In $\mathrm{St}(\Phi, \mathfrak{q})$ let $\hat{U}(\Phi, \mathfrak{q})$ be the subgroup generated by all $x_\alpha(q)$, $\alpha>0$, $q\in\mathfrak{q}$, and $\hat{U}^-(\Phi, \mathfrak{q})$ be the subgroup generated by all $x_\alpha(q)$, $\alpha<0$, $q\in\mathfrak{q}$.

(4.3) THEOREM. *The product map*

$$\hat{U}^-(\Phi, \mathfrak{q}) \times \hat{H}(\Phi, \mathfrak{q}) \times \hat{U}(\Phi, \mathfrak{q}) \xrightarrow{\psi} \mathrm{St}(\Phi, \mathfrak{q})$$

is injective, and $L(\Phi, \mathfrak{q})\cap \operatorname{im}\psi\subset \hat{H}(\Phi, \mathfrak{q})$. *If* ψ *is surjective, then* $\mathfrak{q}\subset \operatorname{rad} A$.

Conversely, suppose $\mathfrak{q}\subset\operatorname{rad} A$ *and assume* $A=\boldsymbol{Z}[(A^*)^2]$ *(resp.* $\boldsymbol{Z}[A^*]$) *if* Φ *is symplectic (resp. nonsymplectic). Then* ψ *is surjective.*

Using these theorems together with known properties [8], [9] of the pairing $\{\ ,\ \}$, one may derive quantitative information about the groups $L(\Phi, A)$ and, in particular, about $K_2(A)$. Some examples are:

COROLLARY. *Let* $m\in\boldsymbol{Z}$, $m>0$, $m\not\equiv 0 \bmod 4$. *Then* $L(\Phi, \boldsymbol{Z}/m\boldsymbol{Z})=0$.

For K_2, this was proved by Milnor [6] using his computation of $K_2(\boldsymbol{Z})$ and results of Mennicke, Bass, Lazard, and Serre on the congruence subgroup problem. More generally:

COROLLARY. *Let* $\mathfrak{O}$ *be a Dedekind domain of characteristic* 0, $0\neq\mathfrak{p}\subset\mathfrak{O}$ *a prime ideal which is unramified over* $p\boldsymbol{Z}=\mathfrak{p}\cap\boldsymbol{Z}$. *If* $\operatorname{rk}\Phi=1$, *assume that* $\mathfrak{O}/\mathfrak{p}\neq \boldsymbol{F}_9$. *Then if* p *is odd,* $L(\Phi, \mathfrak{O}/\mathfrak{p}^n)=0$ *for all* $n\geqq 1$. *Moreover, if* Φ *is nonsymplectic and* $p=2$, $L(\Phi, \mathfrak{p}^{n-1}/\mathfrak{p}^n)$ *is the product of at most* 2^s-1 *cyclic groups of order* 2, *where* $\mathfrak{O}/\mathfrak{p}$ *has cardinality* 2^s.

COROLLARY. *The map* $H_2(\mathrm{SL}(2, \boldsymbol{Z}/2^n\boldsymbol{Z}), \boldsymbol{Z})\to L(A_1, \boldsymbol{Z}/2^n\boldsymbol{Z})$ *is surjective for* $n=1, 2$, *but not for* $n\geqq 3$.

Note. This corollary implies that $\{-1, -1\}\neq 0$ in $L(A_1, \boldsymbol{Z}/4\boldsymbol{Z})$ (cf. [6]). This does not imply a similar result for $K_2(\boldsymbol{Z}/4\boldsymbol{Z})$, since it is only known that the map $L(A_1, \boldsymbol{Z}/4\boldsymbol{Z})\to K_2(\boldsymbol{Z}/4\boldsymbol{Z})$ is surjective.

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