HIGHER DERIVATIONS AND AUTOMORPHISMS OF COMPLETE LOCAL RINGS

BY NICKOLAS HEEREMA

ABSTRACT. This paper begins with a review of those aspects of the theory of higher derivations on fields which form a background for the study of recent uses of higher derivations in automorphism theory of complete local rings. Basic definitions and basic properties of convergent higher derivations on complete local rings are discussed including the concept of convergent rate group of automorphisms, the theory of which is at the present time almost totally undeveloped.

Methods of constructing automorphisms using higher derivations are considered next, particularly in connection with the problem of identifying the factor groups of the higher ramification series of a complete local ring. Recent results on this problem are discussed as well as some possible directions for future research on the topics of this article.

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I. **Introduction.** My purpose in this article is to provide a selective outline of the development of the theory of higher derivations leading to applications in the automorphism theory of complete local rings. As a result certain recent developments in higher derivation theory, except perhaps for casual reference, are outside the scope of this paper, e.g., applications to Galois theory of fields [5], [6], [21], [35], [38] as well as the theory of universal higher derivations and in-

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separability criteria [2], [33]. Proofs, or indications of proof, will be provided now and then to clarify the subject for the general reader.

Let A be an (non-associative) algebra over a field k of characteristic zero and let d be a derivation on A, that is, d is an additive map of A into A which satisfies the product rule, $d(ab) = ad(b) + d(a)b$. We will use Der A to denote the additive group of all derivations on A. The sequence of maps $\{d^i/i!\}$ (all sequences and sums, unless otherwise indicated, are indexed from 0 to ∞) is an example of a higher derivation on A. If A is complete in some topology in which the above sequence of maps converges then $\exp d = \sum (d^i/i!)$ will be an automorphism on A. For example, if k is the field of real numbers and A has finite dimension over k then k is complete in the Euclidean topology and $\exp d$ is a convergent series and hence an automorphism, for all d in Der A. Or, in the general case, if d is associative nilpotent then $\exp d$ contains only a finite number of non-zero terms and is thus an automorphism. Such automorphisms have an important role in the structure theory of Lie algebras over fields of characteristic zero [27, Chapter 9].

II. Basic definitions and basic properties.

(1) DEFINITION. Let $h \supset k$ be (non-associative) rings. A set $D = \{D^{(i)}\}$ of additive maps of k into h which satisfies (i) and (ii) is called a *higher derivation* of k into h.

(i) $D^{(n)}(ab) = \sum \{D^{(i)}(a)D^{(n-i)}(b) \mid i = 0, \cdots, n\}$, $n \geqq 0$.

(ii) $D^{(0)} = \mathrm{Id}$, (identity map on k).

Property (i) is called the *Leibniz rule*. The more familiar Leibniz rule for derivations, namely,

$$d^n(ab) = \sum \binom{n}{i} d^i(a)d^{n-i}(b),$$

allows one to establish (i) in the case of the above example simply by dividing both sides by $n!$

The above definition was first given by F. K. Schmidt in 1937 [10, p. 224] generalizing a concept first introduced, a year earlier, by H. Hasse in connection with the differential calculus of algebraic curves over fields of prime characteristic, [9], [37]. We shall need a pair of results found originally in the Hasse and Schmidt paper [10, pp. 229, 230]. Let $\mathcal{H}(k, h)$ be the set of all higher derivations on the ring k into the overring h. If $k = h$ we abbreviate the notion to $\mathcal{H}(k)$.

(2) PROPOSITION. *Let D be in $\mathcal{H}(k, h)$ where k is a subfield of the field h, and let x be an element of h. If x is transcendental over k, then,*

given $\{u_i | i \geqq 1, u_i \in h\}$, *there exists exactly one extension* D_* *in* $\mathcal{H}(k(x), h)$ *of* D *for which* $D_*^{(i)}(x) = u_i$, $i \geqq 1$. *If* x *is separable algebraic over* k *then* D *extends uniquely to* D_* *in* $\mathcal{H}(k(x), h)$.

One can prove the first assertion simply by constructing D_* on $k[x]$ as dictated by the conditions $D_*^{(i)}(x) = u_i$, the Leibniz rule and additivity. Uniqueness follows from the construction. As in the case of derivations, a higher derivation defined on an integral domain has a unique extension to the field of quotients.

We sketch a nonstandard sort of proof of a slight generalization of the second statement of the proposition, assuming the characteristic of k different from zero. The reason for doing this is that the method of proof generalizes to progressively more complicated situations leading to a convenient method for constructing higher derivations on a complete regular local ring having residue field characteristic different from zero (Theorem 16) [18, p. 28, Theorem 4].

We assume now that characteristic $k = p \neq 0$. A basic property of D in $\mathcal{H}(k, h)$ is the following.

$$D^{(i)}(a^p) = 0 \qquad \text{if } p \nmid i,$$
$$D^{(i)}(a^p) = [D^{(i/p)}(a)]^p \qquad \text{if } p \mid i.$$

The above is proved by $p-1$ applications of the Leibniz rule to $D^{(i)}(a^p)$. It follows then that if n is a positive integer

$$D^{(i)}(a^{p^n}) = 0, \qquad \text{if } p^n \nmid i. \tag{3}$$

Assume now that k_1 is a subfield of h containing k such that k_1/k is separable algebraic. Since k_1/k is separable in the linear disjoint sense and $k_1 = k(k_1^p)$ it follows that, given a basis U for k_1 as a vector space over k, then $U^p = \{u^p | u \in U\}$ is also a basis for k_1 over k. Hence, for any positive integer n, U^{p^n} is a basis for k_1 over k. For a given integer j we choose n so that $p^n > j$. Then, for a in k_1, $a = \sum a_i u_i^{p^n}$, $a_i \in k$, $u_i \in U$, the sum being over a finite subset of U. If $D_* \in \mathcal{H}(k_1, h)$ extends D then

$$D_*^{(j)}(a) = \sum D^{(j)}(a_i) u_i^{p^n} \tag{4}$$

by virtue of (3). The desired result is now obtained from an analysis of (4) which clearly implies that if there is a D_* extending D there is but one such. Moreover, (4) provides a way of constructing D_*. It is a routine matter to verify that the maps $D_*^{(j)}$ constructed using (4) are independent of the choice of n and that $\{D_*^{(j)}\}$ is a higher derivation extending D (for this it is convenient to assume that 1 is in U).

Let $k[[X]]$ be the power series ring in a single indeterminate X over the field k. We consider the group $\mathcal{G}$ of automorphisms on $k[[X]]$

defined by $\mathcal{G}=\{\alpha \mid \alpha-\mathrm{Id}(k[[X]])\subset Xk[[X]],\ \alpha(X)=X\}$. The following bijection between $\mathcal{G}$ and $\mathcal{H}(k)$ was essentially noted by Schmidt [10]. Given D in $\mathcal{H}(k)$ let α_D $(\in\mathcal{G})$ be defined by $\alpha_D(a) = \sum D^{(i)}(a)X^i$ for a in k and $\alpha_D(X)=X$. It is a routine matter to verify that the restriction of α_D to k is an isomorphism which takes a into a power series with constant term a and, hence, that the given conditions determine an element of $\mathcal{G}$. Conversely, if α is in $\mathcal{G}$ and i is a nonnegative integer we define $D^{(i)}(a)$ to be a_i where $\alpha(a)=\sum a_iX^i$. Again, it is a simple matter to show that $D^\alpha=\{D^{(i)}\}$ is in $\mathcal{H}(k)$ and that $\alpha=\alpha_{D^\alpha}$. The bijection $\alpha\to D^\alpha$ induces group structure on $\mathcal{H}(k)$ in the natural way, the group operation " $\circ$ " being given by $D\circ E=F$ where

$$F^{(n)} = \sum \{D^{(i)}E^{(n-i)} \mid i = 0, \cdots, n\}.$$

A portent of things to come; we have a group of higher derivations isomorphic to a group of inertial automorphisms of a complete local ring.

Recently, R. L. Davis obtained a generalization of the Jacobson Galois theory for purely inseparable exponent one field extensions to purely inseparable extensions of arbitrary finite exponent $n+1$ [5], [6], [25]. In this theory Davis uses the finite higher derivation $\{D^{(i)} \mid i=0, \cdots, p^n\}$ as a generalization of derivation and the operation " $\circ$ " to generalize addition of derivations. The closure with respect to pth powers condition of the Jacobson theory appears in the Davis theory in the factors of the upper central series of the group of higher derivations, these being additive groups of derivations.

III. **Construction of higher derivations.** The concept, higher derivation, was originally introduced in order to remove some of the anomalies in the calculus of derivations on fields of characteristic $p\neq 0$ [9], [28]. Nevertheless, higher derivations have important applications in characteristic zero situations, as we shall see. The following result provides considerable information about higher derivations in the latter case.

(5) THEOREM [11, pp. 190–191]. *Let A be an algebra over a field k having characteristic zero. Given a sequence*

$$\{d_i \mid d_i \in \mathrm{Der}\, R,\ i = 1, \cdots, \infty\},$$

the sequence $D=\{D^{(i)}\}$ is in $\mathcal{H}(A)$ where

$$D^{(n)} = \sum \left\{ \frac{d_{i_1}\cdots d_{i_r}}{r!} \,\middle|\, i_1 + \cdots + i_r = n \right\}, \qquad n \geqq 1. \tag{6}$$

Moreover, the correspondence $\{d_i\} \to D$ *is a bijection between the set of all sequences* $\{d_i\}$ *and* $\mathcal{H}(A)$.

This theorem is contained in the above reference for the case in which A is a field, however, the proof given there does in fact serve for the result as stated above.

We see by Theorem 5 that the abundance of higher derivations on A is determined by the abundance of derivations. In 1927, R. Baer showed that the subfields of a field h of characteristic zero which are fields of constants of derivations on h are precisely those subfields k algebraically closed in h [1]. The field of constants of D in $\mathcal{H}(h)$ is $h^D = \{a \in h \mid D^{(i)}(a) = 0,\ i > 0\}$. Since the intersection of subfields of h, each of which is algebraically closed in h, is again algebraically closed in h it follows from the above theorem that the fields of constants of higher derivations on h are again the subfields of h algebraically closed in h.

The similarity between (6) and our first example of a higher derivation suggests that the map exp may have a more substantial connection with higher derivations than the relationship observed in that example. This is the case and that connection as well as the resulting relationship between the Campbell Hausdorff formula and products of higher derivations has been investigated by the writer [22].

An analog of Theorem 5 in the case in which A is a field having characteristic $p, \neq 0$, is given below (Theorem 9). This result is vital to a technique for constructing inertial automorphisms on complete local rings as we shall see presently. But first, an elementary observation which lends some insight into the characteristic p case:

(7) Proposition. *Let h be a field having characteristic $p, \neq 0$, and let k be a subfield of h. If k is perfect then the only higher derivation of k into h is the trivial one* $Q = \{Q^{(i)}\}$, $Q^{(i)}$ *being the zero map for* $i > 0$.

Proof. Let $D = \{D^{(i)}\}$ be in $\mathcal{H}(k, h)$. Since $D^{(1)}(a^p) = pa^{p-1}D^{(1)}(a) = 0$, and $k = k^p$, it follows that $D^{(1)}$ is the zero map θ. If $D^{(i)} = \theta$, for $i < n$, then, by the Leibniz rule, $D^{(n)}$ is a derivation. Thus, by the proof that $D^{(1)} = \theta$ we have $D^{(n)} = \theta$.

The following definition, due to O. Teichmuller, is basic for our purposes. We state it for completeness and refer the reader to the algebra texts for further information [26].

(8) Definition. Let h be a field having characteristic $p, \neq 0$. A subset $\mathcal{S}$ of h is p-independent if the set of all monomials $\{s_1^{i_1} \cdots s_n^{i_n} \mid s_j \in \mathcal{S},\ i_j < p\}$ is linearly independent over h^p. If, in addition, $h^p(\mathcal{S}) = h$ then $\mathcal{S}$ is a p-basis for h.

To illustrate. If $h^p = h$, of course, h has no p-independent subsets. If $h^p \not\subseteq h$ choose $s_1 \in h$, $s_1 \notin h^p$. Then $\{s_1\}$ is a p-independent subset. If $h^p(s_1) \not\subseteq h$ choose s_2 in h, not in $h^p(s_1)$. Then $\{s_1, s_2\}$ is a p-independent subset, etc.

(9) THEOREM [12, p. 131, THEOREM 1]. *Let* $\mathcal{S}$ *be a* p-*basis for the subfield* k *of a field* h *and let* $\{\phi_i\}$ *be a sequence of maps of* $\mathcal{S}$ *into* h. *There is exactly one* D *in* $\mathcal{H}(k, h)$ *such that* $D^{(i)}|_{\mathcal{S}} = \phi_i$ ($D^{(i)}|_{\mathcal{S}}$ *denotes the restriction of* $D^{(i)}$ *to* $\mathcal{S}$).

A proof of Theorem 9, not the original, based on Proposition 2 and the proof of the second part of Proposition 2 can be easily summarized as follows. We note first that $\mathcal{S}$ is an algebraically independent set over the maximal perfect subfield k_∞ of k. We apply the extension, to any number of indeterminates, of the first part of Proposition 2. Thus, there is exactly one D_* in $\mathcal{H}(k_\infty(\mathcal{S}), h)$ such that $D_*^{(i)}|_{\mathcal{S}} = \phi_i$. Now k is separable over $k_\infty(\mathcal{S})$ in the linearly disjoint sense, since $\mathcal{S}$ is a p-basis for both fields. In fact it is easily shown [13, p. 347] that if U is a basis for k as a linear space over $k_\infty(\mathcal{S})$ then U^p is also. Using the argument following Proposition 2 we conclude that each E in $\mathcal{H}(k_\infty(\mathcal{S}), h)$ has a unique extension to $\mathcal{H}(k, h)$ and the proof is complete. We are now in a position to proceed to automorphisms.

IV. **Derivation automorphisms, convergence rate automorphisms and the ramification series.** Let R be a complete local ring with maximal ideal M and residue field $k = R/M$. Let G be the group of automorphisms of R. The M related structure of R suggests a decomposition of G as follows. Let $i \geq 1$.

$$G_i = \{\alpha \in G \mid \alpha(a) - a \in M^i \text{ for } a \text{ in } R\},$$

$$H_i = \{\alpha \in G_i \mid \alpha(a) - a \in M^{i+1} \text{ for } a \text{ in } M\}.$$

The series (10) is called the higher ramification series of G or simply the ramification series. This

$$G_1 \supset H_1 \supset G_2 \supset H_2 \supset \cdots \tag{10}$$

series was first considered by Saunders MacLane who, in 1939, determined the invariant subrings of the groups of (10) as well as the factor groups of successive pairs of subgroups all in the case in which R is the ring of integers of a p-adic field [30].

The M-adic topology on G is the topology obtained by choosing, for each α in G, the cosets $\{G_i\alpha, M_i\alpha \mid i \geq 1\}$ as an open neighborhood basis at α. One sees that G is complete in the M-adic topology.

A higher derivation $D=\{D^{(i)}\}$ on R is said to converge if $\sum D^{(i)}$ is a convergent series of maps. This simply means that there is a sequence $\{n_i\}$ of nonnegative integers for which $\lim n_i=\infty$ and $D^{(i)}(R)\subset M^{n_i}$, $i>0$.

We consider the following subset of $\mathcal{H}(R)$. $\mathcal{H}_c(R)=\{D\in\mathcal{H}(R)\mid \sum D^{(i)}$ converges and $D^{(i)}(M)\subset M^2$, $i>0\}$. The succeeding list of statements is discussed below.

(i) If D is in $\mathcal{H}_c(R)$ then $D^{(i)}(R)\subset M$, $i\geq 1$.

(ii) $\mathcal{H}_c(R)$ is a subgroup of $\mathcal{H}(R)$.

(iii) $\sum D^{(i)}$ is in H_1 for $D=\{D^{(i)}\}$ in $\mathcal{H}_c(R)$.

(iv) $\phi:\mathcal{H}_c(R)\to H_1$ where $\phi(D)=\sum D^{(i)}$ is a group homomorphism. The range of ϕ is called the group of *derivation automorphisms* and is denoted by G_D.

(v) G_D is an invariant subgroup of G.

(vi) Given a sequence $\{n_i\}$ of positive integers such that $\lim n_i=\infty$, the set $\mathcal{H}_{\{n_i\}}=\{D\in\mathcal{H}(R)\mid D^{(i)}(R)\subset M^{n_i},\ D^{(i)}(M)\subset M^{n_i+1}\}$ forms a subgroup of $\mathcal{H}_c(R)$.

(vii) $\phi(\mathcal{H}_{\{n_i\}})=G_{\{n_i\}}$ is an invariant subgroup of G, called a *c.r.* (*convergence rate*) subgroup.

Proofs of (i) and (ii) are found in [**18**, Lemma 1, Theorem 3]. Considering (iii) and (iv), additivity of the $D^{(i)}$ and the Leibniz rule establish that $\sum D^{(i)}$ is an endomorphism. A routine check shows that $D\to\sum D^{(i)}$ is a product preserving map of $\mathcal{H}_c(R)$ into the endomorphism ring of R. But $\phi(Q)=\mathrm{Id}$. Hence $\phi(D)$ is an automorphism and is in H_1 by (i) and the fact that $D^{(i)}(M)\subset M^2$ for $i>0$. With reference to (v), if α is in G and D is in $\mathcal{H}_c(R)$, then $\alpha^{-1}D\alpha=\{\alpha^{-1}D^{(i)}\alpha\}$ is in $\mathcal{H}(R)$ and is, in fact, in $\mathcal{H}_c(R)$ since $\alpha(M^i)=M^i$ for $i\geq 0$. Also, $\alpha^{-1}(\sum D^{(i)})\alpha=\sum\alpha^{-1}D^{(i)}\alpha$. This proves (v) and (vii). The proof of (vi) requires only a (perhaps long) look at the definition of product in $\mathcal{H}(R)$ and the expression

$$\overline{D}^{(n)}=\sum\{(-1)^rD^{(i_1)}\cdots D^{(i_r)}\mid i_1+\cdots+i_r=n,\ i_j>0\}$$

for the nth map of $\overline{D}$ the inverse of D.

We have now two sets of invariant subgroups of G, the ramification series and the c.r. subgroups. Nothing would seem to lead one to suspect that the set of c.r. subgroups is linearly ordered by inclusion. Thus one would expect that the two sets do not coincide. We shall see that this is true in general; however, we shall also see that there are relationships between the two sets. The remainder of this article is concerned primarily with these questions and with the use of derivation automorphisms to evaluate the successive factors of the ramification series (10).

V. Approximating inertial automorphisms with derivation automorphisms. To these ends we consider next the problem of approximating inertial automorphisms with derivation automorphisms in the M-adic topology. We will follow, roughly, the historical development.

The first two techniques are illustrated in the case in which R is a v-ring, that is R is a complete discrete valuation ring having characteristic 0, whereas the residue field of R has characteristic p, $\neq 0$. In this case M is a principal ideal. Let π be a generator of M. If α is in G_i then $\alpha = \mathrm{Id} + \pi^i\alpha^*$ and α is in H_i if and only if $\alpha^*(\pi R) \subset \pi R$. Thus, if α is in H_i, α^* induces a map on k. Since α preserves sums and products we have

$$\alpha^*(a+b) = \alpha^*(a) + \alpha^*(b),$$
$$\alpha^*(ab) = a\alpha^*(b) + \alpha^*(a)b + \pi^i\alpha^*(a)\alpha^*(b).$$

Thus, if $\alpha \in H_i$, α^* induces a derivation δ_α on k, δ_α being the unique map which makes (11) commutative where ξ denotes the natural map of R onto k.

$$\begin{array}{ccc} R & \xrightarrow{\alpha^*} & R \\ \xi \downarrow & & \downarrow \xi \\ k & \xrightarrow{\delta_\alpha} & k \end{array} \tag{11}$$

The homomorphism $\phi_i: G_i \to (k, +)$ and $\psi_i: H_i \to \mathrm{Der}\ k$ where $\phi_i(\alpha) = \xi\alpha^*(\pi)$ and $\psi_i(\alpha) = \delta_\alpha$ are basic to the process of approximating inertial automorphisms in the M-adic topology. The exactness of the following sequences is clear, ϵ being the natural injection.

$$0 \to H_i \xrightarrow{\epsilon} G_i \xrightarrow{\phi_i} (k, +),$$
$$0 \to G_{i+1} \xrightarrow{\epsilon} H_i \xrightarrow{\psi_i} \mathrm{Der}\ k,$$

We consider now the problem of constructing a pre-image in H_i of $\delta \in \mathrm{Der}\ k$ with the aid of ψ_i.

Method 1. We choose d in Der R which induces δ as in diagram (11). We next select $i \geqq (e+1)/(p-1)$, where $pR = \pi^e R$, that is, e is the ramification index of R. We also assume at this point that $p \neq 2$. Let $D = \{\pi^{in}d^n/n!\}$. A check shows that D is in $\mathcal{H}(R)$. In fact, we chose i large enough to insure that D converges. Thus

$$\alpha_D = \sum \frac{\pi^{in}d^n}{n!} = \mathrm{Id} + \pi^i(d + \pi \cdots)$$

is in H_i and, apparently, $\psi_i(\alpha_D) = \delta$.

This technique works fine for many purposes if R is unramified ($e=1$) since in that case every δ in Der k lifts to d in Der R [**13**, p. 349, Theorem 1], $(e+1)/(p-1)=2/(p-1)\leqq 1$ $(p\neq 2)$, and $G_i=H_i$ for $i\geqq 1$. Thus, if R is an unramified v-ring (ring of integers of a p-adic field) the above discussion implies the conclusion that ψ_i induces an isomorphism of H_i/G_{i+1} with Der k. Since $G_i=H_i$, $i\geqq 1$, we have all the factors of (10). As stated earlier this was first done by MacLane who used quite different methods [**30**].

Since, for α in H_i, $\alpha=\alpha_1$, mod H_{i+1} where α_1 is in G_D and by iterating the above process, $\alpha=\alpha_r$, mod H_{i+r}, for α_r in G_D and any $r>0$, it is simply a matter of showing that $\lim_r \alpha_r$ is in G_D to conclude that in this case $G_1=H_1=G_D$.

Method 2. Given δ in Der k we construct d in $\mathcal{H}(k)$ so that $d^{(1)}=\delta$ (Theorem 9) and lift d to D in $\mathcal{H}(R)$ (i.e., $D^{(i)}$ induces $d^{(i)}$ for $i\geqq 1$). Now $\pi^i D=\{\pi^{ni}D^{(n)}\}$ is in $\mathcal{H}_c(R)$ and

$$\alpha_{\pi^i D} = \sum \pi^{ni}D^{(n)} = 1 + \pi^i(D^{(1)} + \pi \cdots)$$

is in H_i and clearly has the property $\psi_i(\alpha_{\pi^i D})=\delta$.

This method is useful if e and p are relatively prime since in that case we have

(i) $\mathcal{H}(k)$ lifts to $\mathcal{H}(R)$[1] and

(ii) $H_i=G_i$, $i\geqq 2$.

As in the unramified case one finds that $H_1=G_D$ and that ψ_i maps onto Der k. Hence ψ_i induces an isomorphism $H_i/G_{i+1}\to$Der k. This fact was demonstrated using other methods by the writer [**17**, p. 538, Theorem 5] in a paper in which it is also shown that the map $\alpha\to\xi(\alpha(\pi)/\pi)$ induces an isomorphism of G_1/H_1 with the group of eth roots of unity in k. The factor group G/G_1 has also been determined in this case [**16**, p. 1208, Corollary 3].

The only result concerning the factors of the ramification series which encompasses all v-rings is due to J. Neggers [**31**, p. 503, Theorem 6].

(12) THEOREM. *Given a v-ring R with ramification e and an integer $i\geqq(e+p)/(p-1)$, then, if α is in G_i, there is a derivation d_α on R such that* $\alpha^*=d_\alpha$, mod M.

Using the techniques of Method 1 α in G_i can be approximated, mod G_{i+1}, by $\beta_i=\sum \pi^{in}d_\alpha^i$. Hence, α can be approximated, mod G_{i+r}, by β_r in G_D for any given integer r. One observes that $\beta=\lim_r \beta_r$ is in G_D and, hence,

[1] Statement (i) was proved by the writer for R unramified [**14**, p. 579, Corollary 1]. The result is easily extended to tamely ramified v-rings.

(13) COROLLARY. *$G_D \supseteq G_i$ for $i \geqq (e+p)/(p-1)$.*

The next corollary is also an immediate consequence of Neggers' Theorem. Let $\mathrm{Der_I} R = \{d \in \mathrm{Der} R / d(\pi) \in \pi R\}$. The homomorphisms

$$\Lambda_i : G_i \to \mathrm{Der}\ R/\pi\ \mathrm{Der}\ R, \quad \Lambda_i' : H_i \to \mathrm{Der_I}\ R/\pi\ \mathrm{Der}\ R$$

and

$$\Lambda_i^* : G_i \to \mathrm{Der}\ R/\mathrm{Der_I}\ R$$

are defined by mapping α into the coset of d_α.

(14) COROLLARY. *Given that $i \geqq (e+p)/(p-1)$ the following are exact.*

$$0 \to G_{i+1} \xrightarrow{\epsilon} G_i \xrightarrow{\Lambda_i} \mathrm{Der}\ R/\pi\ \mathrm{Der}\ R \to 0,$$

$$0 \to H_i \xrightarrow{\epsilon} G_i \xrightarrow{\Lambda_i^*} \mathrm{Der}\ R/\mathrm{Der_I}\ R \to 0,$$

$$0 \to G_{i+1} \xrightarrow{\epsilon} H_i \xrightarrow{\Lambda_i'} \mathrm{Der_I}\ R/\pi\ \mathrm{Der}\ R \to 0.$$

The last sequence above states that the range of ψ_i is precisely the subgroup of those δ in Der k which lift to Der R. Neggers also showed that Der $R = \mathrm{Der_I}$ R if and only if every $\delta \in$ Der k lifts to Der R [**31**, p. 500, Corollary 1].

The factors of the ramification series of an unramified complete regular local ring R (i.e. a power series ring in n indeterminates over a field or an unramified v-ring) have been determined by the writer [**15**, p. 37, Theorems 2.1, 2.2, 2.3]. It is implicit in this work that $H_1 = G_D$ for such a ring R.

The following assertion regarding c.r. groups does not appear in the literature. It is easily proved. One might regard it as an initial result from which one would proceed to study c.r. groups in case R is an unramified complete regular local ring.

(15) PROPOSITION. *Let R be an unramified complete regular local ring and let $\{n_i\}$ be a sequence of positive integers for which $\lim_i n_i = \infty$ and $n_r \leqq n_i + n_{r-i}$ for all r and $i < r$. Then $H_t = G_{\{n_i\}}$ where $t = \min_i n_i$.*

VI. **A more versatile method of approximating.** In order to obtain further results a more versatile method of constructing derivation automorphisms would seem to be needed. Such a method is supplied by the following theorem for which we assume that R is a complete regular local ring in the unequal characteristic case. I. S. Cohen showed that R contains an unramified v-ring V having the property $V + M/M = k$ [**3**, p. 79, Theorem 11].

(16) THEOREM [18, p. 38, THEOREM 4]. *Let $\mathcal{S} \subset V$ be a set of representatives of a p-basis $\bar{\mathcal{S}}$ of k. Given a sequence $\{\phi_i\}$ of maps of $\mathcal{S}$ into R, there is exactly one D in $\mathcal{H}(V, R)$ for which $D^{(i)}|_{\mathcal{S}} = \phi_i$, $i \geq 1$. Also $D^{(i)}(V) \subseteq M^{n_i}$, $i \geq 1$, if and only if $D^{(i)}(\mathcal{S}) \subseteq M^{n_i}$, $i \geq 1$.*

As an immediate corollary we have the following which has been referred to earlier:

(17) COROLLARY. *$\mathcal{H}(k)$ lifts to $\mathcal{H}(V)$.*

The proof of Theorem 16 follows the pattern of argument used to establish Theorem 9 as follows. Let V_0 be the (unique) sub v-ring of V with residue field k_∞, the maximal perfect subfield of k. We note first that the only derivation of V_0 into R is the zero map and, hence, as in the proof of Proposition 7, $\mathcal{H}(V_0, R)$ contains only the trivial higher derivation. Since $\bar{\mathcal{S}}$ is algebraically independent over k_∞, $\mathcal{S}$ is algebraically independent over V_0. Thus, proceeding as in the proof of Theorem 9, we can define D_* on $V_0[\mathcal{S}]$ by the condition $D_*^{(i)}|_{\mathcal{S}} = \phi_i$, $i \geq 1$. A set U, of representatives of a linear basis $\bar{U}$ of k over $k_\infty(\bar{\mathcal{S}})$, is chosen in V. For any preassigned integer n, V is a free module over $V_0[\mathcal{S}]$, modulo $p^n V$, with free generators the set U^{p^n}. This fact implies that D_* has a unique extension to D in $\mathcal{H}(V, R)$, again, as in the proof of Theorem 9 though admittedly the details here are a little messy.

It was proved by I. S. Cohen that if R is as in Theorem 16 then R has the form $S[\pi]$ where $S = V[[X_1, \cdots, X_n]]$ is a power series ring in n indeterminates over V and π is algebraic over S having a minimal polynomial over S of a particular kind called an Eisenstein polynomial [3, p. 92, Theorem 17]. Thus, the potential usefulness of Theorem 16 for the construction of derivation automorphisms on such rings is enhanced by the fact that one knows how D in $\mathcal{H}(V, R)$ extends to a higher derivation defined on R. Theorem 16 has been used in the study of the inertial automorphism group of v-rings with ramification p by the writer [19] and Martin N. Heinzer [24] and for v-rings with ramification $2p$ by Robert D. Davis [4].

VII. **Some applications and some problems.** For R an unramified complete regular local ring the factors H_i/G_{i+1} are all direct sums of copies of Der k, the number of summands depending on i and the dimension of R. The factors G_i/H_i are all zero, or, in the equal characteristic case, they are direct sums of copies of k^+, the additive group of k, the number of summands determined again by i and the dimension of R [15, p. 37, Theorems 2.1, 2.2, 2.3]. Also, as we have observed before $G_D = H_1$, and according to Proposition 15 a large class of c.r. groups, defined by a natural condition on the convergence rate, coincide with the subset $\{H_i\}_{i\geq 1}$ of the ramification series. All of this changes if R is ramified. In particular, if R is a v-ring with ramification p then, in general, G_D is not a term of the ramification series. Moreover, the factors of the ramification series, exhibit a good bit of variety, and are no longer dependent on k (and the dimension of R) [**19**, p. 46, Theorem 2]. As an initial investigation into the c.r. groups M. Heinzer has determined the relationship between G_D, $G_{\{i\}}$ and the ramification series, again, for the v-ring R with ramification p [**24**]. If R is tamely ramified $G_{\{i\}} = G_D$, a fact which is implied by the application of Method II of this paper for approximation of inertial automorphism since all automorphisms constructed by Method II are in $G_{\{i\}}$.

Heinzer found that the relationship between $G_{\{i\}}$ (there denoted G_S) and the other groups was described by a number of different cases [**24**, see table]. Always $H_1 \supset G_{\{i\}} \supset H_2$. Generally, $G_{\{i\}}$ is different from G_D and is not a term of the ramification series.

Robert D. Davis [**4**] carried out the complete analysis of the factors of the ramification series (except G/G_1) for the general v-ring of ramification $2p$, using a modification of the procedures used by the writer in the case $e = p$ [**19**]. Davis found that eight different pairs of factor groups H_1/G_2 and H_2/G_3 occur. However, for $i > 2$, H_i/G_{i+1} is the group of δ in Der k which lifts to Der R (see Neggers result, Theorem 12) and is always given by $\{\delta \in \text{Der } k \mid \delta\xi(\pi^{2p}/p) = 0\}$. Considerable variety was also found in the factors G_i/H_i because of the possible presence of Galois maps, that is automorphisms of finite order. Such maps α always occur in a $G_i - H_i$ gap, i.e., $\alpha \in G_i$, $\alpha \notin H_i$.

Certain conclusions can be drawn from the results discussed above. For example, the problem of obtaining a complete analysis of the factors of the ramification series of all v-rings having a given ramification e, a problem which has been solved for the cases $(e, p) = 1$, $e = p$, and $e = 2p$, is apparently too complex to be manageable in the remaining cases. This is suggested by Davis' results for $e = 2p$. A further classification of v-rings is needed in order to divide the problem into manageable parts.

At this point very little is known about the c.r. groups other than Heinzer's analysis of $G_{\{i\}}$ and Proposition 15. For example, by Proposition 15, the set of groups $G_{\{n_i\}}$ for which $n_r \leqq n_i + n_{r-1}$ is totally ordered by inclusion, if R is unramified and regular. It would be surprising if this were true in general.

Another interesting question is the following: Given distinct sequences $\{n_i\}$ and $\{m_i\}$ of positive integers such that $\lim_i n_i = \lim_i m_i = \infty$. Is there a complete local ring R for which $G_{\{n_i\}} \neq G_{\{m_i\}}$? Also, what property of R determines whether or not $G_{\{n_i\}} = G_{\{m_i\}}$? These questions are of particular interest when $\{n_i\}$ and $\{m_i\}$ differ very little, e.g., $n_i = m_i$ for $i \neq p$ and $n_p = m_p + 1$. The analysis is likely to be quite straightforward if one restricts attention to unramified regular rings only. However, the answers to questions such as these may give rise to interesting classifications of wildly ramified v-rings.

Another result which would be of interest is a generalization of Neggers' theorem (Theorem 12) to complete regular local rings of dimension greater than one.

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