ON THE FREENESS OF ABELIAN GROUPS: A GENERALIZATION OF PONTRYAGIN'S THEOREM[1]

BY PAUL HILL

Communicated by I. N. Herstein, March 9, 1970

Recall that a subgroup H of a torsion-free abelian group G is pure if and only if G/H is again torsion-free. For brevity, call a set S an $f\sigma$-union of its subsets S_λ, $\lambda \in \Lambda$, if each finite subset of S is contained in some S_λ. In this language, Pontryagin's theorem is (equivalent to) the following, where the set-theoretic, not the group-theoretic, union prevails.

THEOREM (PONTRYAGIN). *If the countable, torsion-free abelian group G is the $f\sigma$-union of pure subgroups that are free, then G must be free.*

Pontryagin gave an example that demonstrates that the prefix "$f\sigma$" cannot be deleted from the above theorem; indeed he showed that there exists a torsion-free group of rank 2 that is not free such that each subgroup of rank 1 is free (see, for example, [1, p. 151]). We present the following direct generalization of Pontryagin's theorem obtained by transposing the countability condition.

THEOREM 1. *If the torsion-free abelian group G is the $f\sigma$-union of a countable number of pure subgroups that are free, then G must be free.*

OUTLINE OF PROOF. Let G be an $f\sigma$-union of pure subgroups H_n, $n<\omega$, that are free. Write $H_n = \sum_{i \in I(n)} \{g_i\}$. For simplicity of notation, let μ denote the smallest ordinal having the cardinality of G. We claim that there exist subgroups A_α, $\alpha<\mu$, of G satisfying the following conditions:

(0) $A_0 = 0$.

(1) A_α is pure in G for each $\alpha < \mu$.

(2) $\{A_\alpha, H_n\}$ is pure in G for each $\alpha<\mu$ and each $n<\omega$.

(3) $A_{\alpha+1} \supseteq A_\alpha$ for each α such that $\alpha+1<\mu$.

(4) $A_{\alpha+1}/A_\alpha$ is countable for each α such that $\alpha+1<\mu$.

(5) $A_\alpha \cap H_n = \sum_{i \in I(n,\alpha)} \{g_i\}$ for $\alpha<\mu$ and $n<\omega$, where $I(n, \alpha)$ is a subset of $I(n)$.

AMS 1969 *subject classifications*. Primary 2030; Secondary 2051.

Key words and phrases. Torsion-free abelian groups, Pontryagin's theorem, direct sum of cyclic groups, pure subgroups.

[1] This research was supported by NSF Grant GP-12318.

(6) $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ if β is a limit ordinal less than μ.

(7) $G = \bigcup_{\alpha<\mu} A_\alpha$.

The proof of the existence of the subgroups A_α satisfying conditions (0)–(7) is the same verbatim as that found in our proof of Theorem 1 in [2], so we shall not repeat it here. However, the remainder of the proof of the present theorem differs, in an essential way, from the proof of the theorem just cited. In order to complete the proof here, we need to show that $A_{\alpha+1}/A_\alpha$ is free for each $\alpha<\mu$. We naturally appeal to Pontryagin's theorem since $A_{\alpha+1}/A_\alpha$ is countable. Suppose that S is a finite subset of $A_{\alpha+1}$. There exists $n<\omega$ such that $S \subseteq H_n$ because G is the $f\sigma$-union of the H_n's. Letting B be the smallest pure subgroup of G containing $\{A_\alpha, S\}$, we observe that $B \subseteq A_{\alpha+1} \cap \{A_\alpha, H_n\}$ since $A_{\alpha+1}$ and $\{A_\alpha, H_n\}$ are both pure subgroups of G containing $\{A_\alpha, S\}$. From the relations

$$B/A_\alpha \subseteq \{A_\alpha, H_n\}/A_\alpha \cong H_n/(A_\alpha \cap H_n) \cong \sum_{i \in I(n)-I(n,\alpha)} \{g_i\},$$

we conclude that B/A_α is free. Since B/A_α is a pure subgroup of $A_{\alpha+1}/A_\alpha$ containing $\{A_\alpha, S\}/A_\alpha$ and since S was an arbitrary finite subset of $A_{\alpha+1}$, we see that $A_{\alpha+1}/A_\alpha$ is an $f\sigma$-union of pure subgroups that are free. By Pontryagin's theorem $A_{\alpha+1}/A_\alpha$ is free, and $A_{\alpha+1} = A_\alpha + C_\alpha$ where $C_\alpha \cong A_{\alpha+1}/A_\alpha$ is free. The proof is finished with the observation that $G = \sum_{\alpha<\mu} C_\alpha$.

Two simple applications of the above theorem that demonstrate that the theorem is actually a genuine generalization of Pontryagin's theorem are the next two theorems, which are of interest in their own right.

THEOREM 2. *If the torsion-free abelian group G is the union of a countable chain of pure, free subgroups, then G is free.*

THEOREM 3. *Let m be a cardinal number that is the limit of an ordinary, countable sequence of smaller cardinals. Suppose that the torsion-free abelian group has rank m. If each subgroup of G having rank less than m is free, then G must be free.*

In the statement of Theorem 3, m can be, for example, $\aleph_0$ or $\aleph_\omega$. Combining Theorem 2 with the results of [2] covering the torsion case, we have

THEOREM 4. *If the group G is the union of a countable chain of pure subgroups, then G is a direct sum of cyclic groups if and only if the subgroups belonging to the chain are direct sums of cyclic groups.*

Other applications of these results will appear elsewhere.

REFERENCES

1. L. Fuchs, *Abelian groups*, Akad. Kiadó, Budapest, 1958; republished by Internat. Series of Monographs on Pure and Appl. Math., Pergamon Press, New York, 1960. MR **21** #5672; MR **22** #2644.

2. P. Hill, *Primary groups whose subgroups of smaller cardinality are direct sums of cyclic groups* (to appear).

FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306