ASYMPTOTICS FOR $\square u = m^2u + G(x, t, u, u_t, u_x)$. I. GLOBAL EXISTENCE AND DECAY[1]

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Communicated by M. H. Protter, March 2, 1970

The existence of global solutions to equations of the form

$$\square u = m^2u + G(x, t, u, u_t, u_x), m > 0,$$

can be proved for a wide class of perturbations G. Estimates on the decay of these solutions (i.e. of

$$\|u(t)\|_r = \|u(\cdot, t)\|_r \quad \text{and} \quad \|\dot{u}(t)\|_r = \|u_t(\cdot, t)\|_r$$

as $|t| \to \infty$) which are suitable for the scattering theory of these equations have also been obtained. The results to be summarized here generalize the decay results of Segal [1] for $G(u)$ not only in that more general types of perturbations may be treated but also in the fact that no a priori global existence is required.

1. **Abstract decay result.** Let A^2 denote the selfadjoint realization of $m^2I - \Delta$ on $L^2(E^n)$. The real solution spaces, $H(A, a)$, which are relevant in this work are, for each $a \in R$, the completions of $D(A^a) \oplus D(A^{a-1})$ with respect to the inner product

$$\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)_{A,a} : = (A^a u_1, A^a v_1) + (A^{a-1}u_2, A^{a-1}v_2).$$

The norm of $\binom{u(t)}{\dot{u}(t)} \in H(A, a)$ will be denoted by $|u(t)|_a$ as opposed to the usual L^p-norm $\|u(t)\|_p$, and $G(\cdot, t, u(t), \dot{u}(t), u_x(t))$ will be replaced by $G(t, u(t))$.

A list of assumptions will now be presented leading up to the final result which will be given as a summarizing theorem. To begin, pick r and a in such a way that

$$\text{(I)} \qquad \|u(t)\|_r, \|\dot{u}(t)\|_r \leqq \text{Const} \, | u(t) |_a,$$

so that the continuity of $\|u(t)\|_r$ and $\|\dot{u}(t)\|_r$ follows from that of

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AMS 1969 *subject classifications*. Primary 3557, 3537, 3516; Secondary 8135.

Key words and phrases. Klein-Gordon equation, nonlinear perturbation, Cauchy problem, global existence, temporal decay, coupled nonlinear inequalities, Sobolev inequalities.

[1] This work was presented in part as an invited address to the Research Colloquium of the Conference on Mathematical Theory of Scattering held in Flagstaff, Arizona, July—August 1969. Research supported in part by the National Science Foundation (NSF GP 13627).

$|u(t)|_a$. Assume G is sufficiently smooth and has suitable growth properties so that (Sobolev inequalities will give)

$$\text{(II)} \qquad \|A^b G(t, u(t))\|_q \leqq g \text{ Const} |u(t)|_a^{2\alpha} \|u(t)\|_r^{\beta} \|\dot{u}(t)\|_r^{\gamma}$$

for $1 + r^{-1} = p^{-1} + q^{-1}$ as well as $q = 2$ for some $b \leqq a$, and

$$\text{(III)} \quad \|A^{a-1} G(t, u(t))\|_2 \leqq g \text{ Const} |u(t)|_a^{2(\alpha''-1/2)} \|u(t)\|_r^{\beta''} \|\dot{u}(t)\|_r^{\gamma''}.$$

Assumption (II) is required twice with perhaps different values of q (i.e. also for q' with corresponding exponents α', β', γ'). For the particular choice of b in (II) consider the fundamental solutions of the Klein-Gordon equation (i.e. $G \equiv 0$) $E_{t,b}$ and $F_{t,b}$ defined as the regular distributions whose Fourier transforms are

$$(\xi^2 + m^2)^{-(b+1)/2} \sin t(\xi^2 + m^2)^{1/2} \quad \text{and} \quad (\xi^2 + m^2)^{-(b+1)/2} \cos t(\xi^2 + m^2)^{1/2}$$

respectively. For the p and p' determined by (II) suppose that the known results [2] on the decay of $E_{t,b}$ and $F_{t,b}$ give $\|E_{t,b}\|_p \leqq \text{Const}(1 + |t|)^{-\rho}$ and $\|F_{t,b-1}\|_{p'} \leqq \text{Const}(1 + |t|)^{-\sigma}$ for all $t \in \boldsymbol{R}$ then the following consistency condition must be satisfied by all the exponents:

$$\text{(IV)} \qquad \begin{aligned} &\max(\rho, \beta\epsilon + \gamma\delta) > 1, \qquad \min(\rho, \beta\epsilon + \gamma\delta) \geqq \epsilon, \\ &\max(\sigma, \beta'\epsilon + \gamma'\delta) > 1, \qquad \min(\sigma, \beta'\epsilon + \gamma'\delta) \geqq \delta, \\ &\beta''\epsilon + \gamma''\delta > 1, \end{aligned}$$

where ϵ and δ are the anticipated decay rates of $\|u(t)\|_r$ and $\|\dot{u}(t)\|_r$ respectively. Finally, the Cauchy data at some finite time t_0 are chosen smooth enough so that the corresponding solution of the Klein-Gordon equation, $u_0(t)$ decays at least as fast as that desired for the perturbed equation, or more precisely,

$$\text{(V)} \quad \|u_0(t)\|_r \leqq x_0(1 + |t|)^{-\epsilon} \quad \text{and} \quad \|\dot{u}_0(t)\|_r \leqq \dot{x}_0(1 + |t|)^{-\delta}$$

for all $t \in \boldsymbol{R}$, where x_0, $\dot{x}_0$ are locally bounded functions of t_0 which can be made small by reducing the size of the Cauchy data.

ABSTRACT THEOREM. *Suppose that the (integrated form of the) equation* $\Box u = m^2u + G(x, t, u, u_t, u_x)$ *has a unique solution* $\binom{u(t)}{\dot{u}(t)} \in H(A, a)$ *over some interval* I *containing* t_0, *such that* $\binom{u(t)}{\dot{u}(t)}$ *is continuous from* $I \to H(A, a)$. *If assumptions* (I)–(V) *are satisfied and either*

(i) $\alpha + \beta + \gamma$, $\alpha' + \beta' + \gamma'$, $\alpha'' + \beta'' + \gamma'' < 1$,

(ii) $\alpha + \beta + \gamma$, $\alpha' + \beta' + \gamma'$, $\alpha'' + \beta'' + \gamma'' > 1$ *and* x_0, $\dot{x}_0$ and $y_0 = \sup_{t \in R} |u_0(t)|^2$ *sufficiently small or g sufficiently small or*

(iii) $\alpha, \beta, \cdots, \beta'', \gamma''$ *arbitrary and* x_0, $\dot{x}_0$, y_0 *and* g *sufficiently small,*
then the solution can be extended to all $t \in R$ *and* $\|u(t)\|_r = O(|t|^{-\epsilon})$, $\|\dot{u}(t)\|_r = O(|t|^{-\delta})$ *and* $|u(t)|_a = O(1)$ *as* $|t| \to \infty$.

The proof follows in part along the same lines as that of Segal [**1**] but ultimately a technical result concerning coupled sets of nonlinear inequalities is required.

2. **Examples.** Many different perturbations fall within the scope of the above result (and slight variants of it) by making specific choices of the parameters (results stated for $n=3$). In all of the cases listed below the local existence of the solution as assumed in the Abstract Theorem can be obtained from the various hypotheses by means of the general results of Segal [**3**] in this direction.

THEOREM 1. (Cf. [**1**, Corollary 4.4B, p. 491].) *Suppose* $G(x, t, u, u_t, u_x) = G(u)$ *where* $G \in C^2(R)$, *is real-valued and* $|d^jG/d\lambda^j(\lambda)| \leqq g|\lambda|^{\beta-j}$ *for all* λ *and* $j=0, 1, 2$, *with* $\beta \geqq 3$. *If the Cauchy data* $\binom{u_1}{u_2} \in H(A, 2)$ *are sufficiently smooth so that* (V) *is satisfied then the* (*integrated form of the*) *equation* $\square u = m^2u + G(u)$ *has a unique global solution with* $\|u(t)\|_\infty = O(|t|^{-3/2})$, $\|\dot{u}(t)\|_\infty = O(|t|^{-\delta})$ *where* δ *is arbitrary but* <1 *and* $|u(t)|_2 = O(1)$ *as* $|t| \to \infty$, *provided* g *or* $x_0 + \dot{x}_0 + y_0$ *is sufficiently small.*

The proof is obtained by taking $b=2$, $q=1$ and $1 < q' < 1/\delta$.

THEOREM 2. *Suppose* $G(x, t, u, u_t, u_x) = G(u_t)$ *where* $G \in C^3(R)$, *is real-valued and* $|d^jG/d\lambda^j(\lambda)| \leqq g|\lambda|^{\beta-j}$ *for all* λ *and* $j=0, 1, 2, 3$ *with* $\beta \geqq 3$. *If the Cauchy data* $\binom{u_1}{u_2} \in H(A, 3)$ *is sufficiently smooth so that condition* (V) *is satisfied then* (*the integrated form of*) *the equation* $\square u = m^2u + G(u_t)$ *has a unique global solution* $\binom{u(t)}{\dot{u}(t)} \in H(A, 3)$ *with* $\|u(t)\|_\infty = O(|t|^{-\epsilon})$ *where* $\epsilon = \epsilon(\beta) = 3/2$ *for* $\beta > 7/2$ $\epsilon(\beta) < 3(\beta-1)/5$ *for* $3 \leqq \beta \leqq 7/2$, $\|\dot{u}(t)\|_\infty = O(|t|^{-\delta})$ *where* $\delta < 1$ *and* $|u(t)|_3 = O(1)$ *as* $|t| \to \infty$ *provided that* g *or* $x_0 + \dot{x}_0 + y_0$ *is sufficiently small.*

The proof is obtained by taking $b=2$, $q = \max(1, 5(\beta+3/2)^{-1})$ and $\delta < q'^{-1} < \frac{1}{2}(\beta - 1/\delta)$. In both of the above theorems the basic estimate corresponding to (II) and (III) is, for $w \in D(A^2)$,

$$\|A^2G(w)\|_q \leqq g \text{ Const}\|w\|_\infty^{\beta-2/q}\|A^2w\|_2^{2/q}, \qquad 1 \leqq q \leqq 2.$$

THEOREM 3. *Suppose* $G(x, t, u, u_t, u_x) = \sum_{k=0}^{3} G^k(u)u_{x_k}$ $(x_0 = t)$ *where for each* $k=0, 1, 2, 3$, $G^k \in C^3(R)$, *is real-valued* $|d^jG^k/d\lambda^3(\lambda)| \leqq g|\lambda|^{\alpha^k - j}$ *for all* λ *and* $j=0, 1, 2$ *with* $\alpha^k \geqq 2$ *and* $|d^3G^k/d\lambda^3(\lambda)| \leqq g|\lambda|^{\alpha_3^k}$ *for all* λ *with* $0 \leqq \alpha_3^k < \infty$. *If the Cauchy data* $\binom{u_1}{u_2} \in H(A, 3)$ *are* *sufficiently smooth so that the condition* (V) *is satisfied, then* (*the integrated form of*) *the equation* $\Box u = m^2u + \sum_{k=0}^{3} G^k(u)u_{x_k}$ *has a unique global solution* $\binom{u(t)}{\dot u(t)} \in H(A, 3)$ *with* $\|u(t)\|_\infty = O(|t|^{-1})$, $\|\dot u(t)\|_\infty = O(|t|^{-5/6})$ *and* $|u(t)|_3 = O(1)$ *as* $|t| \to \infty$, *provided that either* g *or* $x_0 + \dot x_0 + y_0$ *is sufficiently small.*

The proof is obtained by taking $b = 2$, $q = q' = 6/5$.

THEOREM 4. *Suppose* $G(x, t, u, u_t, u_x) = g\ G(x, t)u$ *where* G *is real-valued for each* t, $G(\cdot, t)$ *is an element of the Sobolev space* $W^{2,p}(E^3)$ *for* $p = 1$ *and* ∞ (*or equivalently* $1 \leqq p \leqq \infty$), $\|G(\cdot, t)\|_{2,p}$ *is continuous for* $t \in \boldsymbol{R}$, $\|G(\cdot, t)\|_{2,1}$ *uniformly bounded in* t *and* $\|G(\cdot, t)\|_{2,\infty} = O(|t|^{-3})$ *as* $|t| \to \infty$. *If the Cauchy data* $\binom{u_1}{u_2} \in H(A, 3)$ *are sufficiently smooth so that condition* (V) *is satisfied then the* (*integrated form of the*) *equation* $\Box u = m^2u + g\ G(x, t)u$ *has a unique global solution* $\binom{u(t)}{\dot u(t)} \in H(A, 3)$ *with* $\|u(t)\|_\infty = O(|t|^{-3/2})$, $\|\dot u(t)\|_\infty = O(|t|^{-\delta})$, $\delta < 1$, *and* $|u(t)|_3 = O(1)$ *as* $|t| \to \infty$ *provided that* g *is sufficiently small.*

The proof is obtained by taking $b = 2$, $q = 1$ and $1 < q' < 1/\delta$. The hypotheses on G are suggested by those that would be obtained from taking $G(x, t) = (v(x, t))^2$ where v is a solution of the Klein-Gordon equation with very smooth Cauchy data (i.e. the first variational equation of $\Box u = m^2u + gu^3$).

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