COMMUTING VECTORFIELDS ON OPEN MANIFOLDS[1]

BY J. WOLFGANG SMITH

Communicated by Norman Steenrod, March 24, 1969

Let M be an open orientable differentiable n-manifold. More precisely, we will take M and vectorfields over M to be of class C^∞. A nonzero vectorfield X on M will be called *nonrecurrent* if the 1-dimensional foliation associated with X is regular (see [4, Chapter I]) and admits no compact leaves. The notation $H^p(M; Z)=Q$ shall mean that the p-dimensional singular integral cohomology of M is trivial or admits no torsion of order 2, depending on whether p is even or odd, respectively.

THEOREM 1. *Let X be a nonrecurrent vectorfield on M and let $A\subset M$ be relatively compact. When $H^{n-1}(M; Z)=Q$ there exists a vectorfield Y on A such that X, Y are linearly independent and commute.*

THEOREM 2. *When $H^{n-1}(M; Z)=Q$ every relatively compact subset of M submerges in the plane.*

For $n>4$ Theorem 2 is implied by a result of I. M. James and E. Thomas (quoted as Theorem 8.6 in [5]). Moreover, we note that the cohomological triviality condition is crucial to both Theorems 1 and 2. A very simple example shows this in the case of Theorem 1: Let M be Euclidean 3-space with a point 0 removed and let $X=\partial/\partial r$, where r denotes distance to 0. Let S denote the unit sphere centered at 0 and let $\pi\colon M\rightarrow S$ denote radial projection. There exist relatively compact subsets $A\subset M$ such that $\pi(A)=S$. A vectorfield Y on A which commutes with X induces then a vectorfield $\bar{Y}$ on S such that $\bar{Y}$ pulls back to Y under $d\pi$. Moreover, if (X, Y) are linearly independent, $\bar{Y}$ must be nonzero, showing that the conclusion of Theorem 1 does not hold in this case. It is also possible to display examples of open orientable C^∞-manifolds M with relatively compact $A\subset M$ which do not submerge in the plane. We may take M to be the punctured real projective space of dimension 5, for instance. It is known [5, p. 201] that this space does not submerge in the plane. But obviously M admits relatively compact subsets A which are in fact diffeomorphic to M.

In this note we shall derive Theorems 1 and 2 from results established in [6]. First a few definitions: If F is a regular orientable p-

[1] Research supported in part by National Science Foundation Grant GP-6648.

dimensional foliation on M, we let M/F denote the quotient space obtained by identifying points of M belonging to the same leaf of F. Regularity of F implies that M/F can be regarded as a differentiable manifold. It will be orientable, but in general non-Hausdorff. Let χ_F denote its Euler class (the algebraic sign of χ_F being determined by a choice of orientation). The foliation F is said to *extend on A* (A being a subset of M) provided there exists an orientable $(p+1)$-dimensional foliation $\hat{F}$ on A with $F\subset\hat{F}$.

THEOREM A. *An orientable regular foliation F on M with $\chi_F=0$ extends on relatively compact subsets of M.*

When M/F is Hausdorff, this conclusion follows easily by classical obstruction theory. In the general case it does not. An essential ingredient in the proof is a triangulation theorem contributed by J. R. Munkres, which appears to be also of independent interest (see [**6**, Appendix]). The next step is

LEMMA B. *Let F be a regular 1-dimensional foliation on M without compact leaves. The natural projection π: $M\to M/F$ induces then an isomorphism between the respective singular homology groups.*

We refer to [**6**] for proofs of Theorem A and Lemma B. Combining these results and noting that χ_F has order 2 when n is even (see [**3**, p. 41]) we obtain what will be needed:

EXTENSION THEOREM. *Let F be a regular orientable 1-dimensional foliation on M without compact leaves. If $H^{n-1}(M; Z)=Q$, then F extends on relatively compact subsets of M.*

We proceed to establish Theorem 1. Let X be a nonrecurrent vectorfield on M and $A\subset M$ an open relatively compact subset. Let B denote a compact neighborhood of A and let F denote the foliation associated with X. If $H^{n-1}(M; Z)=Q$, then by the Extension Theorem there exists an orientable 2-dimensional foliation $\hat{F}$ on B with $F\subset\hat{F}$. Using a Riemannian metric on M one obtains a differentiable field Z of unit vectors on B such that Z is orthogonal to X and (X, Z) span $\hat{F}$. But this implies that $[X, Z]=aX+bZ$ where a, b are differentiable functions on B. Let $Y=\alpha X+\beta Z$, where α, β are smooth functions on B. An easy calculation shows that the condition $[X, Y]=0$ is equivalent to the differential equations

$$\langle X, d\beta\rangle + \beta b = 0 \tag{1}$$

$$\langle X, d\alpha\rangle + \beta a = 0 \tag{2}$$

where $\langle\ ,\ \rangle$ denotes the inner product. It remains to be shown that this system admits a solution on A with $\beta > 0$. But this can be accomplished by the classical theory of characteristics for first order equations (see for instance [1, Chapter 2]). Equations (1) and (2) both have the integral curves of X as their characteristics. The problem is thus reduced to integrating the ordinary differential equations

$$d\beta/ds + \beta b = 0 \tag{1*}$$

$$d\alpha/ds + \beta a = 0 \tag{2*}$$

along the integral curves of X, where s denotes a parameter. To construct a solution, we cover A by a finite sequence of "tubes" $T_1, \cdots, T_r$; each T_i being determined by a local cross-section S_i to the characteristics. More precisely, S_i is taken to be a closed disc of dimension $n-1$ smoothly imbedded in B which meets each integral curve of X in at most one point (which is possible by nonrecurrence of X). The tube T_i is then taken to be the set of all points $x \in B$ which can be connected to S_i by an integral curve of X lying entirely in B. We note that each T_i is a closed subset of B. One can prescribe β, α on S_1, and this determines β, α on T_1 by integrating equations (1*) and (2*), respectively. We note also that $\beta > 0$ on S_1 implies $\beta > 0$ on T_1. Let us assume that β, α have been determined on $T_1, \cdots, T_j$ for some $j < r$ so as to be consistent on the intersections and such that $\beta > 0$. The values of β, α are then prescribed on $S_{j+1} \cap T_1 \cap \cdots \cap T_j$, which constitutes a closed subset of S_{j+1}. The given functions can be smoothly extended to S_{j+1}, preserving $\beta > 0$. Integrating Equations (1*) and (2*) over T_{j+1} with the given initial values gives an extension of β, α to T_{j+1}. The construction is thus completed in r steps.

To prove Theorem 2 one first observes that since M is open it submerges in the real line (by Theorem 4.7 in M. Hirsch [2]). Given a Riemannian metric, such a submersion determines a nonrecurrent vectorfield X on M (i.e. the gradient of the submersion). If

$$H^{n-1}(M; Z) = Q$$

and A is an open relatively compact subset of M, then by Theorem 1 (or equally well, by the Extension Theorem) there exists a vectorfield Y on A such that X, Y are linearly independent. By Theorem B in Phillips [5] this implies that A submerges in the plane.

References

1. R. Courant and D. Hilbert, *Methods of mathematical physics*. Vol. II, Interscience, New York, 1962.

2. M. Hirsch, *On embedding differentiable manifolds in Euclidean space*, Ann. of Math. (2) **73** (1961), 566–571.

3. J. W. Milnor, *Lectures on characteristic classes*, Mimeographed notes, Princeton University, Princeton, N.J., 1957.

4. R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. No. 22, 1957.

5. A. Phillips, *Submersions of open manifolds*, Topology **6** (1966), 171–206.

6. J. W. Smith, *Extending regular foliations*, Ann. Inst. Fourier (Grenoble) (to appear).

OREGON STATE UNIVERSITY, CORVALLIS, OREGON 97331