ON SUMMABILITY FIELDS OF CONSERVATIVE OPERATORS

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Let $B[c]$ denote the Banach algebra of all bounded linear operators on c, the set of convergent sequences. By a conservative operator we mean a member of $B[c]$. If $T \in B[c]$ and if there exists an infinite matrix $A = (a_{nk})$ such that $Tx = Ax$ for each $x \in c$, then T is called a conservative matrix. (By $Tx = Ax$ we mean $(Tx)_n = (Ax)_n \equiv \sum_k a_{nk}x_k$ for each $n \in I^+$, the set of positive integers.) Let Γ denote the subalgebra of $B[c]$ of all conservative matrices. If $T \in \Gamma$, its summability field, denoted by c_T, is taken to be the set $\{x \in s: Tx \in c\}$, where s denotes the set of all sequences. This raises the following question: How can one define the summability field c_T for an arbitrary T in $B[c]$? In other words, which sequences should one distinguish as being the set that a conservative operator sums?

One viewpoint is to consider how T acts on c_0, the maximal subspace of c consisting of those sequences which converge to 0. The restriction of T to c_0 is always representable by a matrix. In other words, if T' denotes the restriction of T to c_0, then there is an infinite matrix B so that $T'x = Bx$ for each $x \in c_0$. Surely, the summability field of T' is the set $c_B = \{x \in s: Bx \in c\}$. We now note that if T is a conservative matrix, say A, then A also represents the restriction of T to c_0, i.e. $A = B$. Thus, it seems reasonable to require that $c_T \supseteq c_B$ for any conservative operator T, where B is the matrix representing the restriction of T to c_0. Since the unit sequence $e = (1, 1, 1, \cdots)$ need not belong to c_B, even though Te always belongs to c, we cannot, in general, take $c_T = c_B$. However, since e is the only basis element of c that B might not sum, we propose that c_T be defined as

$$c_T = c_B \oplus e,$$

where $\oplus$ denotes the linear span of the sets c_B and e. The purpose then of this announcement is to report how the properties of c_T defined above for $T \in B[c]$ compare with the well-known properties of c_T for $T \in \Gamma$.

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Further evidence supporting the choice of the above definition of c_T is found in the following remarks. An FK space (i.e., locally convex Fréchet sequence space with continuous coordinates [5]) which contains c is called conull if e belongs to the weak closure of c_0; otherwise, it is called coregular. Those $T\in\Gamma$ for which c_T is conull are characterized by belonging to the kernel of the only nonzero multiplicative linear functional on Γ, denoted by χ. (χ is defined in §1 below.) The problem of extending the concept of conullity from Γ to all of $B[c]$ was dealt with in an earlier paper [1]. It was shown there that there is exactly one subalgebra of $B[c]$, denoted by Ω, which properly includes Γ, and that χ has a unique extension, denoted by ρ, to a nonzero multiplicative linear functional on Ω. The kernel of ρ, therefore, became the natural definition for conullity in $B[c]$. We note here that the kernel of ρ is precisely the set of those conservative operators T for which c_T, as defined above, is a conull FK space.

1. **Further definitions and terminology.** For each $k\in I^+$, let e^k denote the sequence having one in the kth coordinate and zeros elsewhere. If $x\in c$, $\lim x$ means $\lim_i x_i$. On $B[c]$ we have the functions

$$\chi(T) = \lim Te - \sum_k \lim Te^k$$

and

$$\chi_i(T) = (Te)_i - \sum_k (Te^k)_i$$

for each $i\in I^+$. (The functions χ, χ_i are defined in [4]. See also [1].) It was pointed out in [4] that Γ is precisely the set of those conservative operators T for which $\chi_i(T)=0$ for every $i\in I^+$. The set of those conservative operators for which $\lim_i \chi_i\,(T)$ exists is denoted by Ω. The structure of the subalgebras Γ and Ω was studied in [1]. As was observed there, we may write each $T\in B[c]$ as follows:

$$Tx = (\lim x)v + Bx \quad (\text{for } x \in c)$$

where $v=\{\chi_i(T)\}$ and B is the matrix representing the restriction of T to c_0. This relationship between T, v and B will be denoted by $T\sim(v, B)$. We remark here that if $T\in\Omega$, then $B\in\Gamma$, while if $T\notin\Omega$, then $v\in m\backslash c$ (where m denotes the set of bounded sequences), $B: c_0\to c$, and $e\notin c_B$.

2. **The summability field c_T and its dual space c_T'.** Let $T\in B[c]$, say $T\sim(v, B)$. Since c_B is always an FK space [5, p. 228], so is c_T. Define a functional α on c_T as follows: If $T\in\Omega$ take $\alpha\equiv 0$, while if

$T\notin\Omega$, let $\alpha(e)=1$ and $\alpha(x)=0$ for each $x\in c_B$. Then α is a continuous linear functional on c_T, i.e. $\alpha\in c_T'$. Now let

$$S = \{x \in s: \alpha(x)\cdot v + Bx \in c\}.$$

Since B: $m\rightarrow m$ and $v\in m$ we always have $m_B\supset S$, where

$$m_B = \{x \in s: Bx \in m\}.$$

But B is continuous as a map from m_B into m [5, Corollary 5, p. 204], $S=B^{-1}(c\oplus v)$, and $c\oplus v$ is a closed subspace of m; hence, S is a closed subspace of m_B. Thus, S is an FK space [5, p. 203]. Now, by defining d_T, the domain of T, to be the set $\{x\in s: Bx\in s\}$, a straightforward application of [5, Theorems 5 and 6, p. 230] reveals that each $f\in S'$, the dual space of S, has the representation

$$\begin{aligned} f(x) &= \delta\cdot\alpha(x) + d_0\cdot\lim_n\left(v_n\cdot\alpha(x) + \sum_k b_{nk}x_k\right) \\ &\quad + \sum_n d_n\left(v_n\cdot\alpha(x) + \sum_k b_{nk}x_k\right) + \sum_k \beta_k x_k, \end{aligned} \tag{1}$$

where $\beta=\{\beta_k\}\in s$, $\sum|d_n|<\infty$, and d_0 and δ are scalars, and α is the functional defined above. Moreover, since the kernel of α is precisely c_B we see that $S=c_T$. Finally, since $c_B\supset c_0$ we see that c_T is coregular whenever $T\notin\Omega$. We summarize these remarks in the following theorem.

THEOREM 1. *For any conservative operator T, c_T is an FK space and the most general continuous linear functional on c_T is given by equation* (1). *Moreover, if $T\notin\Omega$, then c_T is coregular.*

3. **Some properties of c_T.** A well-known result for matrix summability fields is that they cannot be properly contained between c and m [5, Problem 31, p. 231]. This property is not retained by c_T, as the following example illustrates.

Let B be defined by the set of equations

$$\begin{aligned} b_{nn} &= (-1)^{n+1}, && n = 1, 2, \cdots, \\ b_{2n,2n-1} &= 1, && n = 1, 2, \cdots, \\ b_{nk} &= 0, && \text{otherwise.} \end{aligned}$$

Then $c_B=c_0\oplus y$, where $y=\{1, 0, 1, 0, \cdots\}$ and hence if we set $v=\{0, 1, 0, 1, \cdots\}$, then $T\sim(v, B)$ defines a conservative operator such that $c_T=c\oplus y$.

The following theorem sheds some light on the structure of c_T when it is a subset of m.

THEOREM 2. *If $c_T \subset m$, either $c_T = c$ or c is a maximal closed subspace of c_T.*

PROOF. If $T \in \Omega$, then $c_T = c_B$ and $B \in \Gamma$, and so $c_T = c$ whenever $c_T \subset m$. Suppose $T \notin \Omega$ and $c_T \subset m$. Then the topology of c_T is the same as that of m [5, Corollary 1, p. 203] and so c is closed in c_T since it is closed in m and they have the same topology.

Let $f \in c_T'$ with $f(e^k) = 0$ for each $k \in I^+$. Then the representation (1) yields

$$\beta_k = -d_0 b_k - \sum_n d_n b_{nk},$$

where $b_k = \lim_n b_{nk}$. Substituting this back into (1) and recalling that $c_T \subset m$ and $\sum |d_n| < \infty$ we obtain

$$\begin{aligned} f(x) &= \delta \cdot \alpha(x) + d_0 \cdot \lim_n \left(v_n \cdot \alpha(x) + \sum_k b_{nk} x_k \right) \\ &\quad + \sum_n d_n v_n \alpha(x) - \sum_k d_0 b_k x_k. \end{aligned} \tag{2}$$

If we now also assume that $f(e) = 0$, then, by letting $x = e$ in (2) and using the fact that $\alpha(e) = 1$, we see that

$$0 = \delta + d_0 \cdot \lim_n \left(v_n + \sum_k b_{nk} \right) + \sum_n d_n v_n - \sum_k d_0 b_k.$$

Since

$$\chi(T) = \lim_n \left(v_n + \sum_k b_{nk} \right) - \sum_k b_k$$

and since we may add a convergent sequence to $\{v_n\}$ without changing c_T, we see that we may assume $\chi(T) = 0$. (Indeed, set $v_n' = v_n - \chi(T)$ for each $n \in I^+$ and let $T' \sim (v', B)$ to obtain $c_{T'} = c_T$ and $\chi(T') = 0$.) It follows that $\delta + \sum_n d_n v_n = 0$. Letting

$$\Lambda(x) = \lim_n \left(v_n \alpha(x) + \sum_k b_{nk} x_k \right) - \sum_k b_k x_k$$

we see that every functional which vanishes on c has the form $f(x) = d_0 \cdot \Lambda(x)$, and so the proof is complete.

Another well-known result in summability is that a conservative matrix A is compact (i.e., $\sum_k |a_{nk}|$ converges uniformly with respect to n) if and only if it is coercive (i.e., $m \subset c_A$). Theorem 3 below will show that this result extends to conservative operators.

LEMMA. *If $T \notin \Omega$, then $m \not\subset c_T$.*

PROOF. Suppose that $m \subset c_T$. Then B: $c_0 \to c$ and $c_B \oplus e \supset m$; hence, the proof of Schur's Theorem [3, p. 17] shows that $\sum_k |b_{nk}|$ converges uniformly with respect to n, that is, B is a compact operator on m.

Let $z^1, z^2, \cdots$ be a bounded set in c. Then

$$v \cdot \alpha(z^1) + Bz^1,\ v \cdot \alpha(z^2) + Bz^2,\ \cdots$$

is a subset of c. Let

$$v \cdot \alpha(y^1) + By^1,\ v \cdot \alpha(y^2) + By^2,\ \cdots$$

be any subsequence. Since B is a compact operator on m, By^1, $By^2, \cdots$ has a convergent subsequence in m, say $Bx^1, Bx^2, \cdots$. Since $m \subset c_T$, $v \cdot \alpha(x)$ is also a compact operator on m, so $v \cdot \alpha(x^1)$, $v \cdot \alpha(x^2), \cdots$ also has a convergent subsequence in m, say $v \cdot \alpha(u^1)$, $v \cdot \alpha(u^2), \cdots$. Thus,

$$v \cdot \alpha(u^1) + Bu^1,\ v \cdot \alpha(u^2) + Bu^2,\ \cdots$$

is a subset of c and converges, that is, T is a compact operator. Since c has a Schauder basis, T is the uniform limit of operators with finite dimensional range. But each such operator belongs to Ω. (For example, if T has one dimensional range, say $Tx = f(x) \cdot u$, where $u \in c$ and $f \in c'$, then

$$\chi_i(T) = \left(f(e) - \sum f(e^k)\right) \cdot u_i$$

converges. The general case follows from this one.) It follows from the fact that Ω is closed in $B[c]$ that $T \in \Omega$. This proves the lemma.

THEOREM 3. *$m \subset c_T$ if and only if T is compact.*

PROOF. Suppose T is compact. Then, as was pointed out in the proof of the lemma, $T \in \Omega$. It follows that B is a compact conservative matrix and hence coercive. Thus, $m \subset c_B = c_T$.

Conversely, if $m \subset c_T$, then $T \in \Omega$. Hence, $c_T = c_B$ and B is a compact conservative matrix. Since $v \in c$, $v \cdot \alpha(x)$ is also a compact operator on c. Thus, T is compact.

For each $T \in B[c]$, say with $T \sim (v, B)$, we have already observed that there is associated a continuous linear functional α on c_T so that

$$c_T = \{x \in s : v \cdot \alpha(x) + Bx \in c\}.$$

Thus, we may define

$$\lim_T x = \lim_n \left(v_n \cdot \alpha(x) + \sum_k b_{nk} x_k \right)$$

for $x \in c_T$. It is clear that $\lim_T \in c_T'$. We say that T satisfies the *consistency property* if the conditions $S \in B[c]$, $c_S \supset c_T$ and $\lim_T x = \lim_S x$ on c always imply that $\lim_T x = \lim_S x$ on c_T. We remark here that the procedure developed by Mazur [2] for use with conservative matrices can readily be adapted to conservative operators, and yields the following consistency-type result.

THEOREM 4. *A conservative operator T satisfies the consistency property if and only if c is dense in c_T.*

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