BOOK REVIEWS

Markov processes and potential theory, by R. M. Blumenthal and R. K. Getoor. Monographs in Pure and Applied Mathematics, Academic Press, New York, 1968. x+313 pp. $15.00.

This is a very important book. It will certainly be, in the years to come, the basic text for teaching Markov processes, and the basic reference on this subject.

The theory of time continuous Markov processes consists nowadays of three main fields of activity. First we have much work done on time continuous Markov chains, that is on processes with a very simple discrete state space, under weak hypotheses allowing very discontinuous sample functions. Quite to the opposite, we find the theory of diffusion processes, concerned with nice sample continuous processes on manifolds. Blumenthal and Getoor's book is devoted to the "middle" field, that of all "reasonable" sample right continuous Markov processes on locally compact spaces, along the lines of Hunt's fundamental papers,[1] that is, emphasizing the interplay with ideas and methods borrowed from potential theory. It is quite remarkable that, while so many people were doing research under Hunt's influence, this book is the first one to cover entirely the subject matter of Hunt's papers. Special thanks are due to the authors for having made accessible to all mathematicians the most cryptic parts of MP2 and MP3.[1]

Let us first describe the contents of the book, starting with the chapters concerned with Hunt's work. Chapter I contains the background of Markov processes: Markov property, transition function, strong Markov property, standard processes and Hunt processes, measurability of hitting times, and the basic fact that processes with a Feller transition semigroup are Hunt processes. Chapter II contains the core of MP1: excessive functions and their behavior on sample paths, exceptional sets, fine topology. However, the main balayage theorem is excluded from this chapter and given in Chapter III with the "relative theory of potentials": subordinate semigroups and resolvents, subprocesses, special sets, multiplicative functionals. That is, Chapter III contains a much modernized version of MP2. Finally, the contents of MP3 are given in Chapter VI: dual processes,

[1] *Markov processes and potentials*. I, II, III, Illinois J. Math. 1(1957), 44–93; 316–396; 2 (1958), 151–213. We refer to these papers below as MP1, MP2, MP3.

potentials of measures, capacities. Duality appears here (as it did in MP3) as a purely analytical device: L-times and time reversal are not considered.

On the other hand, the book gives a fairly complete account of the theory of multiplicative and additive functionals (only multiplicative functionals with values in $[0, 1]$ are considered, with the slight inconvenience that h-path processes are thus excluded). Multiplicative functionals appear in Chapter III with the relative theory of potentials: the 1-1 correspondence between them and subordinate semigroups, and their strong Markov property, are proved in this chapter. Additive functionals[2] are the main subject of Chapter IV, which contains the representation theory of excessive functions as potentials of natural additive functionals, and the classification of excessive functions. Chapter V (one of the highlights of the book) starts with a section on reference measures and applications to the fine topology, but its main aim is the proof of several recent theorems on continuous additive functionals: fine supports, Motoo's "Radon-Nikodým theorem" for continuous functionals, balayage of functionals, local times (including Boylan's theorem on the joint continuity of local time in space and time), time changes, and processes with identical hitting distributions (the Blumenthal-Getoor-McKean theorem). Connections between additive functionals and measures are given in Chapter VI under the duality hypotheses. Finally, one must mention a section of notes and comments, mainly historical with the notable exception of an extra theorem on pp. 299–302, a bibliography, index of notation and subject index.

It may be worthwhile to mention explicitly some subjects that *are not included* in this book. First of all, all the line in probabilistic potential theory that stems from Doob's work on the behavior of excessive functions at infinity (Fatou type theorems, h-paths, and boundary theory) is consistently omitted—quite understandably, since it would deserve a whole book by itself. I have already mentioned the omission of time reversal. Nothing is said about the general problem of piecing together Markov processes, a subject which seems to be generally underestimated. The only serious omission, in my opinion, is that of Lévy systems, a theory which belongs really to the domain of this book, and certainly is the main recent contribution to the subject. Of course, it is easy to understand why it was not

[2] More precisely, additive functionals of (X, M), where X is the process and M is one fixed multiplicative functional. This is a nice and useful generalization, but the authors warn us that at a first reading, or for teaching, it is advisable to take $M=1$ throughout.

included, since the definitive papers of the Japanese school on Lévy systems appeared while Blumenthal and Getoor's treatise was being written, but I hope the authors will be able to add this subject to some further edition.

This book certainly is excellent for teaching (and probably also for self-teaching). The authors generally explain the purpose and scope of their definitions and results, give many examples and counterexamples, exercises and comments. They have carefully avoided the Bourbakist trend of grouping together all theorems concerning one given subject in their logical order, regardless of motivation: all auxiliary theorems and technical results are introduced precisely at the place they are needed (sometimes they are not formally stated, and appear only as remarks and comments). All this is excellent pedagogy, and gives the book a pleasant undogmatic look. However, it has as its counterpart some difficulty in using the book discontinuously: if you want to localize some known theorem on stopping times or resolvents, you may at first have some trouble if you do not know the book well. Also, if you want to quote an isolated statement from the book, it is advisable to check from the beginning of the section on whether some hypothesis has been added.[3] Here is an amusing example of pedagogical versus logical order: the resolvent of the basic semigroup appears early, on p. 41; then we find the idea of a resolvent subordinate to the basic one on p. 115, but we must wait till p. 252 before learning what a resolvent (in general) is! It is only fair to say that in this case, as in many others, the subject index is perfectly efficient.

The authors have not used typography to stress important results: all kind of statements and proofs are printed alike: the reader must therefore keep in mind that *theorems* are generally more important than *propositions*, and of course than *remarks*. However, it is one of the most unfortunate features of this theory that you must remember a tremendous number of "important" things, and that unstressed results can be quite crucial. Take for instance p. 284, line 13, the remark that "also (4.1) implies that condition (4.1) of chapter IV holds." I can translate this in my own language as follows: "if all excessive functions are lower semicontinuous, then the basic semigroup is special standard" (that is, its canonical family of σ-fields has

[3] One hypothesis almost hidden is the fact that the semigroup of a standard process (in the authors' sense) carries Borel functions into Borel functions (pp. 45 and 20). This is not at all disturbing to the common reader, but the technician must know that such an hypothesis is assumed.

no times of discontinuity). I consider this as a very striking result on the role of semicontinuity in potential theory (by the way, it is due to Blumenthal and Getoor themselves, and printed here for the first time). As it stands here, it will a.s. pass unnoticed.

The methods used in the book are fairly "elementary": I mean that the kind of measure theory needed for reading the book is standard, except for Choquet's capacity theorem (which is clearly stated, but not proved), and that no complicated martingale theory is used: the classical theorems concerning sample function behavior, and estimates for the supremum of a martingale, are the only prerequisites. It is also interesting to know that the general theory of stochastic processes (separability, etc.) is not necessary to read this book.

The notation is generally very good, quite consistent (except that μf^{-1} for the image of the measure μ by f is unclassical, and no better than the classical notation). The authors should also be praised for sailing a middle course between a too restrictive idea of a Markov process, and heavy axiomatics. Their §3 of Chapter I is a short, but sufficient, introduction to axiomatics. The printing is very clear, and I could not find any misprints, though it is the reviewer's duty to find some.

Besides improving many proofs of known theorems, Blumenthal and Getoor proved several important new results in connection with the writing of this book. Most important, in my opinion, are the theorem on the "perfectness" of continuous additive functionals (Theorem 2.1 of Chapter VI, p. 205), and above all Theorem 6.1 of Chapter III, p. 136, which is the hard part of the extension of Hunt's balayage theorem to all standard processes. Generally speaking, the authors have tried to prove for all standard processes the results known only for Hunt processes. This extension is true most of the time, but sometimes quite difficult.

To end this review, I would like to emphasize the difficulty of writing a book on this subject, and consequently the remarkable achievement of this one. In this theory you start in a desert, learning first what a Markov process is, which kind of Markov process you want to consider, then you are taught how to play with kernels, resolvents, stopping times, capacities, martingales, réduites . . . and generally your guide (book or lecturer) gets exhausted just at the time things become interesting. The authors here have been so successful in managing economically the crossing of that desert that less than one third of the book is spent in generalities, and that more than one third consists of material that you cannot find anywhere else (neither in Dynkin's treatise, nor in my own Lecture Notes volume). Thus every mathematician interested in time continuous Markov processes should know this book.

P. A. MEYER

Algebraic topology by Edwin H. Spanier. McGraw Series in Higher Mathematics, McGraw-Hill, New York, 1966. xiv+528 pp. $15.00.

This book is designed to be a text for a year or longer graduate course in algebraic topology. It also is a reference work for the subject. Like most books that serve both of these purposes, this one has its good and bad properties. Before stating some of the good properties, let me remark that overall I feel the book is an excellent one and the best book on the subject to date. One of the best features of this book over others is the excellent choice of topics covered. All of them are important topics which should be known by those students who wish to work in algebraic topology or to use algebraic topology in other fields. The organization is excellent and well thought-out; it is not a collection of individual topics as some books are. Furthermore, the notation is good and conforms to current usage. The fine set of exercises also helps the organization of the book as some of them lead the reader into material to be covered later.

Some of the disadvantages of using this book as a text stem from its reference work attributes. For example, in some places the topics are covered too thoroughly and this means the reader can become bogged down in some theorems of only technical interest. It also means that the chapter on homology, the most basic concept of algebraic topology, does not begin until p. 154. To counteract the abundance of material and to make sure which are the most important theorems, the reader is advised to read carefully the first paragraph of each section which is a guide to the important results. Another problem is that the book is quite difficult for many students to read, especially on their own. Its use should be accompanied by lectures which have lots of examples and which point out which results proven in the book can be skipped. Another somewhat negative observation is that the book basically only contains ideas that were developed before the mid 1950's. However, a reader who has mastered this book is in a good position to tackle later developments such as K-theory and applications of algebraic topology to differential topology. If the reader is interested in a book which avoids the abundance of material and which is easier to read on one's own, he should try to obtain the lecture notes for 2 courses the author gave at the University of Chicago in 1955 from which this book developed.