A NOTE ON WEAKLY COMPLETE ALGEBRAS[1]

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Fix a commutative noetherian ring R with unit and an ideal I in R. P. Monsky and G. Washnitzer have developed the notion of a weakly complete finitely generated algebra over (R, I) [1], [2]; we include a definition in §2. They have used these "w.c.f.g. algebras" to construct a p-adic De Rahm cohomology for nonsingular varieties defined over fields of characteristic p [1]. It is important for their theory that w.c.f.g. algebras are noetherian; we prove this fact here. Our proof attempts to follow the well-known proof that power series rings over R are noetherian. At one point we need a general lemma concerning modules over polynomial rings; §1 deals with this.

1. Let $R'=R[X_1, \cdots, X_n]$. The degree of a polynomial $f\in R'$ is denoted by ∂f. If S is a finitely generated free R'-module with a fixed basis, identify S with $(R')^m$, and for $f=(f_1, \cdots, f_m)\in S$, define $\partial f=\operatorname{Max} \partial f_i$.

LEMMA. *Let M be a submodule of S, S as above. Then M has a finite number of generators g_α so that any $g\in M$ may be written $g=\sum a_\alpha g_\alpha$ with $a_\alpha\in R'$ and $\partial a_\alpha \leqq \partial g-\partial g_\alpha$.*

PROOF. Let $R^*=R[X_0, X_1, \cdots, X_n]$, $S^*=(R^*)^m$. For each $f=(f_1, \cdots, f_m)\in S$, with $\partial f=d$, write $f^*=(f_1^*, \cdots, f_m^*)\in S^*$, where $f_i^*=X_0^d f_i(X_1/X_0, \cdots, X_n/X_0)$. Let M^* be the (homogeneous) submodule of S^* generated over R^* by $\{g^*\mid g\in M\}$. For the desired generators take any finite set of $g_\alpha\in M$ so that the g_α^* generate M^*. In fact, if $g\in M$, we may write $g^*=\sum A_\alpha g_\alpha^*$, and by homogeneity we may assume $A_\alpha\in R^*$ is of degree $=\partial g^*-\partial g_\alpha^*=\partial g-\partial g_\alpha$. Replacing X_0 by 1 in this equation shows that $g=\sum a_\alpha g_\alpha$, $a_\alpha=A_\alpha(1, X_1, \cdots, X_n)$, and $\partial a_\alpha\leqq \partial A_\alpha=\partial g-\partial g_\alpha$.

2. DEFINITION [2, §2.1]. An R-algebra A is a w.c.f.g. algebra over (R, I) if it satisfies the following two conditions:

(i) $\bigcap_{i=0}^\infty I^iA=0$. We therefore identify A with its image under the natural map $A\to A^\infty=\operatorname{proj}\lim_i A/I^iA$.

(ii) There are elements $x_1, \cdots, x_n$ in A so that for any $y\in A$ there are polynomials $p_d(X_1, \cdots, X_n)\in I^d[X_1, \cdots, X_n]$ and a

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constant $\lambda>0$ with $\partial p_d \leqq \lambda(d+1)$ and $y=\sum_{d=0}^{\infty} p_d(x_1, \cdots, x_n)$.

THEOREM. *A w.c.f.g. algebra over (R, I) is noetherian.*

PROOF. With A, $x_1, \cdots, x_n$ as in the definition, let $R' = R[X_1, \cdots, X_n]$. Take any generators $z_1, \cdots, z_m$ for the ideal I, and form the power series ring $R'[[Z_1, \cdots, Z_m]]$. Denote by $S^{(d)}$ the elements of $R'[[Z_1, \cdots, Z_m]]$ which are homogeneous of degree d in $Z_1, \cdots, Z_m$. $S^{(d)}$ is a free R'-module with the natural base of monomials in $Z_1, \cdots, Z_m$. For $f^{(d)} \in S^{(d)}$ the degree $\partial f^{(d)}$ is defined with respect to this basis, as in §1. Each $F \in R'[[Z_1, \cdots, Z_m]]$ has a unique representation $F=\sum_{d=0}^{\infty} f^{(d)}$ with $f^{(d)} \in S^{(d)}$; we will use this notation without comment.

Let $R'' = \{ \sum f^{(d)} \in R'[[Z_1, \cdots, Z_m]] \mid \partial f^{(d)} \leqq \lambda(d+1)$ for some constant $\lambda \}$. Sending X_i to x_i and Z_i to z_i defines a homomorphism from R'' onto A, so it will suffice to prove that R'' is noetherian.

Suppose, then, that J is an ideal of R''. Let

$$M^{(d)} = \left\{ f^{(d)} \in S^{(d)} \;\Big|\; \sum_{j=d}^{\infty} f^{(j)} \in J \text{ for some } f^{(d+1)}, \cdots \right\},$$

and let M be the homogeneous ideal of $R'[Z_1, \cdots, Z_m]$ generated by all the $M^{(d_i)}$. Take a finite number of generators $h_i \in M^{(d_i)}$ for the ideal M, and choose an integer N greater than all the d_i. For each $k<N$ take a finite number of

$$Q_{j,k} = \sum_{d=k}^{\infty} q_{j,k}^{(d)} \in J$$

whose leading terms $q_{j,k}^{(k)}$ generate $M^{(k)}$ as an R'-module. Likewise take $Q_j=\sum_{d=N}^{\infty} q_j^{(d)} \in J$ so that the $q_j^{(N)}$ generate $M^{(N)}$: here, however, apply the lemma of §1 to $M^{(N)} \subset S^{(N)}$, and choose the $q_j^{(N)}$ to satisfy the conditions of that lemma. By our choice of N, $M^{(d)} = S^{(d-N)} M^{(N)}$ for $d \geqq N$; from this and the lemma we deduce:

(*) Any $g^{(d)} \in M^{(d)}$, $d \geqq N$, may be written in the form
$g^{(d)} = \sum a_j q_j^{(N)}$ with $a_j \in S^{(d-N)}$ and $\partial a_j \leqq \partial g^{(d)}$.

We can now finish the proof by showing that $\{Q_{j,k}, Q_j\}$ generates J. Let $G \in J$. By first subtracting an R''-linear combination of the $Q_{j,k}$, we may assume that $G=\sum_{d=N}^{\infty} g^{(d)}$. We must find $T_j=\sum_{d=0}^{\infty} t_j^{(d)}$ and a constant λ so that $G=\sum T_j Q_j$ and $\partial t_j^{(d)} \leqq \lambda(d+1)$ for all j, d. Choose any $t_j^{(0)}$ such that $g^{(N)} = \sum t_j^{(0)} q_j^{(N)}$. Take μ so large that $\partial t_j^{(0)} \leqq \mu$ for all j, and so that $\partial q_j^{(N+d)} \leqq \mu(d+1)$ for all j, d. Let $\lambda = 2\mu$.

Suppose that $t_j^{(0)}, \cdots, t_j^{(d-1)}$ have been found satisfying $\partial t_j^{(k)}$

$\leqq \lambda(k+1)$ and $g^{(N+k)} = \sum_j \sum_{i=0}^{k} t_j^{(i)} q_j^{(n+k-i)}$ for $0 \leqq k \leqq d-1$. By (*) we can find $t_j^{(d)}$ so that $g^{(N+d)} - \sum_j \sum_{i=0}^{d-1} t_j^{(i)} q_j^{(N+d-i)} = \sum t_j^{(d)} q_j^{(N)}$ and $\partial t_j^{(d)} \leqq \mathrm{Max}\ \{\partial g^{(N+d)},\ \partial(t_h^{(i)}) + \partial(q_h^{(N+d-i)});\ 1 \leqq i \leqq d-1, \text{ all } h\}$. Fortunately this last number is $\leqq \lambda(d+1)$, and so, defining the $t_j^{(d)}$ inductively, the required T_j are found.

REMARK. Other "weakly complete" algebras could be defined by specifying less restrictive growth conditions. For example, the last condition of the definition could be changed to read: $\partial p_d \leqq \lambda(d^\rho + 1)$, where ρ is some real number $\geqq 1$, either fixed or depending on the element y. The above proof shows that all of these algebras are noetherian.

BIBLIOGRAPHY

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