AN AMENABLE GROUP WITH A NONSYMMETRIC GROUP ALGEBRA[1]

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Let G be a discrete group, $l_1(G)$ the group algebra of G. Symmetry of $l_1(G)$ has been considered in [1], [3]. Groups containing a free subgroup on two or more generators are the only groups found to have nonsymmetric group algebras, and in each case the groups found to have symmetric algebras are in the family of amenable groups. In this note we present an example of an amenable group with a nonsymmetric group algebra.

LEMMA 1. *Let G be a group generated by a and b such that S, the semigroup generated by a and b, is free and such that $cd^{-1}=dc^{-1}$ for $\{c, d\} = \{a, b\}$. Then $l_1(G)$ is nonsymmetric.*

PROOF. We will show that $l_1(G)$ is not symmetric by showing that the involution is not hermitian. In particular, we will show that $-i\in\mathrm{sp}(x)$ where

$$x = a + ib + a^{-1} - ib^{-1}$$

(we do not distinguish between G and its canonical image in $l_1(G)$). This is accomplished by defining a θ in $m(G)$, the bounded complex valued functions on G, such that $\|\theta\| = 1 = \theta(e)$ and such that

$$\theta^v[(x + ie)g] = 0$$

for each $g\in G$, where $\theta\rightarrow\theta^v$ is the mapping of $m(G)$ onto $l_1(G)^*$.

Let $S' = S\cup S^{-1}\cup\{e\}$, and define $\theta(g) = 0$ for $g\in G\sim S'$.

We divide the elements of G into the five disjoint sets; S, S^{-1}, $S_1 = a^{-1}bS \cup \{a^{-1}b\}$, $S_2 = ab^{-1}S^{-1} \cup \{ab^{-1}\}$ and $S_3 = G\sim(S\cup S^{-1}\cup S_1\cup S_2)$. Let $A = \{a, b, e, a^{-1}, b^{-1}\}$. Direct computations yield $Ag\cap S'\neq\emptyset$ if and only if $g\notin S_3$ or $g = e$.

Note now that if $g\in S_3$ and $g\neq e$ then

$$\begin{aligned}[\text{support } (x + ie)g] \cap S' &= Ag \cap S' \\ &= \emptyset,\end{aligned}$$

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and hence,

$$\theta^v[(x+ie)g] = \sum_{t\in G} [(x+ie)g](t)\theta(t)$$
$$= 0.$$

We enumerate the elements of S' as follows:

$$s_0 = e; \qquad \text{for } n = 0, 1, 2, \cdots, s_{2n+1} = as_n,$$
$$s_{2n+2} = bs_n, s_{-(2n+1)} = a^{-1}s_{-n},$$
$$s_{-(2n+2)} = b^{-1}s_{-n}; \qquad \text{for } n = 1, 2, 3, \cdots,$$
$$t_n = a^{-1}s_{2n}, t_{-n} = as_{-2n}.$$

Let $y=x+ie$. We have

$$\theta^v(y) = \theta(s_1) + i\theta(s_2) + \theta(s_{-1}) - i\theta(s_{-2}) + i\theta(s_0),$$
$$\theta^v(yt_1) = i\theta(s_1) + \theta(s_2),$$
$$\theta^v(yt_{-1}) = \theta(s_{-2}) - i\theta(s_{-1}).$$

Set $\theta(s_0)=\theta(e)=1$. The following system of equations has a solution such that $0<|\theta(s_i)|\leqq 1$ for $-2\leqq i\leqq 2$;

$$\theta(s_1) + i\theta(s_2) + \theta(s_{-1}) - i\theta(s_{-2}) = -i,$$
$$i\theta(s_1) + \theta(s_2) = 0,$$
$$-i\theta(s_{-1}) + \theta(s_{-2}) = 0.$$

With these values of $\theta(s_i)$, $-2\leqq i\leqq 2$, we have $\theta^v(y)=0$, $\theta^v(yt_1)=0$, and $\theta^v(yt_{-1})=0$.

Now

$$\theta^v(ys_1) = 1 + i\theta(s_1) + \theta(s_2) + i\theta(s_4)$$

and

$$\theta^v(yt_2) = i\theta(s_3) + \theta(s_4).$$

The equations

$$\theta(s_3) + i\theta(s_4) = -1 - i\theta(s_1),$$
$$i\theta(s_3) + \theta(s_4) = 0,$$

have a solution such that $0<|\theta(s_i)|\leqq 1$ for $i=3$, 4. Thus, with these values for $\theta(s_3)$ and $\theta(s_4)$, we have $\theta^v(ys_1)=0$ and $\theta^v(yt_2)=0$.

For k an arbitrary positive integer,

$$\theta^v(ys_k) = \theta(s_{2k+1}) + i\theta(s_{2k+2}) + i\theta(s_k) + (-1)^{k-1}\theta(s_{k'})$$

and

$$\theta^v(yt_{k+1}) = i\theta(s_{2k+1}) + \theta(s_{2k+2}),$$

where k' is the largest integer less than or equal to $(k-1)/2$.

Hence, assuming $\theta(s_j)$ has been assigned for $0 \leqq j \leqq 2k$ and that $0 < |\theta(s_j)| \leqq 1$, the following equations have a solution such that $0 < |\theta_{2k+1}| \leqq 1$ and $0 < |\theta_{2k+2}| \leqq 1$;

$$\theta(s_{2k+1}) + i\theta(s_{2k+2}) = -i\theta(s_k) + (-1)^{k-1}\theta(s_{k'}),$$
$$i\theta(s_{2k+1}) + \theta(s_{2k+2}) = 0.$$

Therefore, we can define $\theta(s)$ for each $s \in S$ such that $\theta^v(ys') = 0$ for each $s' \in S \cup S_1$. Similarly, we can define $\theta(s)$ for each $s \in S^{-1}$ so that $\theta^v(ys') = 0$ for each $s' \in S^{-1} \cup S_2$. Hence, $\theta^v(ys) = 0$ for each $s \in G$.

In [2] an example is given of an amenable group that contains a free subsemigroup on two generators. We will construct a similar example in which the generators of the free subsemigroup satisfy a certain relation within the group.

Let $L(R)$ be the group of ordered pairs of real numbers (u, v), $u \neq 0$, with multiplication defined by the formula

$$(u, v)(u_1, v_1) = (uu_1, uv_1 + v).$$

THEOREM 2. *Let* $s = (a, 1)$ *and* $t = (-a, 1)$, $0 < a \leqq 1/2$, *be elements of* $L(R)$. *Let* G_0 *be the group generated by* s *and* t. *Then* G_0 *is amenable and* $l_1(G_0)$ *is nonsymmetric.*

PROOF. $L_1 = \{(1, b) \mid b \in R\}$, the commutator subgroup of $L(R)$, is Abelian. Thus, $G_0 \cap L_1$ is Abelian. Also

$$G_0/G_0 \cap L_1 \cong G_0L_1/L_1 \subset L(R)/L_1,$$

and hence $G_0/G_0 \cap L_1$ is Abelian. Therefore G_0, the extension of an Abelian group by an Abelian group, is amenable.

To show that $l_1(G_0)$ is nonsymmetric, we first note that $st^{-1} = (-1, 2) = ts^{-1}$. Hence $cd^{-1} = dc^{-1}$ if $\{c, d\} = \{s, t\}$.

An induction argument shows that each word w in S is of the form

$$w = ((-1)^\epsilon a^n, \sum_{i=0}^{n-1} p(i)a^i)$$

for some positive integer n, where $\epsilon = 0, 1$ and $p(i) = \pm 1$ for $0 \leqq i \leqq n-1$.

If for two distinct words w_1 and w_2 in S we have $w_1 = w_2$, then, by cancelling like factors on the left, we may assume we have distinct words w_3 and w_4 such that $sw_3 = tw_4$. But the second term of sw_3 is

$$1 + a + \sum_{i=2}^{n} p(i)a^i > 1$$

with $p(i) = \pm 1$ and some integer n, and the second term of tw_4 is

$$1 - a + \sum_{j=2}^{m} q(j)a^j < 1$$

with $q(j) = \pm 1$, $2 \leqq j \leqq m$ and some integer m. Hence $w_1 \neq w_2$ for distinct words w_1 and w_2 in S. Therefore, the semigroup generated by s and t is free.

Lemma 1 now applies.

REFERENCES

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