A COMPLETE ELEMENTARY PROOF THAT HILBERT SPACE IS HOMEOMORPHIC TO THE COUNTABLE INFINITE PRODUCT OF LINES[1]

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1. **Introduction.** In this paper we give a complete and self-contained proof that real Hilbert space, l_2, is homeomorphic to the countable infinite product of lines, s; symbolically $l_2 \sim s$. We assume only that the reader understands the material of a first year course in topology: for example, elementary notions of complete metric spaces, product spaces, continuity, homeomorphisms, and the Tietze Extension Theorem. While the treatment is elementary, the arguments are not necessarily simple. The reader should be prepared to draw figures and verify continuity statements.

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We define l_2 as the space of square summable sequences of reals—that is, $l_2 = \{(x_i)_{i>0} \mid x_i$ is a real number and $\sum x_i^2 < \infty\}$. For $x, y \in l_2$, the distance between $x = (x_i)_{i>0}$ and $y = (y_i)_{i>0}$ is defined by $d(x, y) = [\sum (x_i - y_i)^2]^{1/2}$. It is easy to verify that under such a distance function, l_2 is a complete metric space.

We define s as the space of all sequences $(x_i)_{i>0}$ of reals with the usual product topology. Thus $s = \prod_{i>0} R_i$, where for each i, R_i is the space of reals. A basis for the topology of s is the collection of sets of the form $\prod_{i>0} U_i$, where U_i is an open subset of R_i and for all but a finite number of the i's, $U_i = R_i$.

Since for each positive integer i, the real line R_i is homeomorphic with the open interval $(-1, 1) = I_i^0$, we may alternatively write $s = \prod_{i>0} I_i^0$. We find this formulation convenient in §3.

At times (§§7 and 9) we need to regard s as a complete metric space. A convenient complete metric to use is $d(x, y) = \sum(\min[1/2^i, |x_i - y_i|])$. An alternate complete metric is the traditional Fréchet metric $d(x, y) = \sum |x_i - y_i| / [2^i(1 + |x_i - y_i|)]$. It is immaterial, for our purposes, which we use, so let us suppose for simplicity that we use the first.

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The question as to whether l_2 is homeomorphic to s was raised by Fréchet [10] in his book, *Les êspaces abstraits* (1928), and also by Banach [5] in his book, *Théorie des opérations linéaires* (1932). In an added footnote in [5], Banach erroneously attributed to Mazur a negative answer to the question. More recently, Kadec, Bessaga, Pelczynski, and others have studied homeomorphisms between various infinite dimensional Banach and Fréchet spaces. The work up to about 1964 is summarized by Bessaga in [6]. In the direction of proving that $l_2 \sim s$, Bessaga and Pelczynski [9] had proved that $l_2 \sim (l_2 \times s)$ and that l_2 was homeomorphic to the countable product of copies of itself. Kadec [11], [12] proved that all separable infinite dimensional Banach spaces are homeomorphic. In 1966, Anderson [1] proved that $l_2 \sim s$. This was done by using results in [2] together with the result that $l_2 \sim (l_2 \times s)$. Based on an earlier Bessaga and Pelczynski result cited in [6], the results of Kadec [11] and Anderson [1] established that all separable infinite dimensional Fréchet spaces are homeomorphic to each other.

In the present paper, the result that $l_2 \sim s$ appears in §3 as Theorem 1.1, but its proof makes use of lemmas and theorems from later sections. The procedure is summarized in §10. §11 lists open questions.

2. **Further definitions and notation.** We let $X \backslash Y$ denote the set of all elements of X not in Y.

Let Z denote the set of positive integers. For $\alpha \subset Z$, we let $\alpha' = Z \backslash \alpha$. We let $s^\alpha = \prod_{i \in \alpha} R_i$ and l_2^α denote the space of sequences $(x_i)_{i \in \alpha}$ of reals for which $\sum_{i \in \alpha} x_i^2 < \infty$ with $d(x, y) = (\sum_{i \in \alpha} (x_i - y_i)^2)^{1/2}$.

The following two propositions are well known and easy to prove.

LEMMA 2.1. *If α is a nonempty subset of Z for which α' is nonempty, then $s \sim s^\alpha \times s^{\alpha'}$. If α is infinite, then $s_2^\alpha \sim s$.*

LEMMA 2.2. *If α is a nonempty subset of Z for which α' is nonempty, then $l_2 \sim l_2^\alpha \times l_2^{\alpha'}$. If α is infinite, then $l_2 \sim l_2^\alpha$.*

We use the convention that if α is an infinite subset of Z, then propositions concerning l_2 and s are considered as applicable to l_2^α and s^α.

If $\alpha \subset Z$, let τ_α denote the projection of l_2 or s onto l_2^α or s^α. If $\alpha = \{i\}$, we may write τ_α as τ_i.

A set K in l_2 or s is *deficient in the ith direction* if $\tau_i(K)$ consists of a single element. A set K in l_2 or s is *infinitely deficient* if for some infinite set $\alpha \subset Z$, $\tau_i(K)$ consists of a single element for each $i \in \alpha$. In this case we also say that K is *deficient with respect to α*.

We use the term map to denote a continuous function. For f a map of X into Y and $K \subset X$, the notation $f|K$ denotes the map f

restricted to the domain K. We use Id to denote the identity homeomorphism.

Let G be an open covering of a space X, Y be a subset of X, and f be a map of Y into X. We say that *f is limited by G* if for each $p \in Y$, there exists a $g_p \in G$ such that $p \in g_p$ and $f(p) \in g_p$. The covering G provides a measure as to how close f is to the identity. If X is a metric space, the *mesh* of G is the least upper bound of the diameters of the elements of G.

Suppose $f_1, f_2, \cdots$ is a sequence of maps such that the limit of $f_1, f_2 \circ f_1, \cdots$ exists. We denote this limit by $\lim(f_i \circ \cdots \circ f_2 \circ f_1)_{i>0} = L\prod_{i>0} f_i$ and call it the infinite left product of the f_i.

In §5 we define *invertible isotopies* and adopt notation to be used in working with isotopies. In §7 we define an *invertibly continuous family of invertible isotopies*.

We use S_1 to denote the set of all points of l_2 whose distance from the origin is 1—that is, $S_1 = \{x \in l_2 \mid \sum x_i^2 = 1\}$.

We use E^i to denote $\{x \in l_2 \mid x_j = 0 \text{ if } j > i\}$. Then $\bigcup E^i$ is the union of all finite dimensional coordinate planes of l_2. Sometimes $\bigcup E^i$ is called infinite dimensional Euclidean space. We note that $l_2 \setminus \bigcup E^i = \{x \in l_2 \mid x_i \neq 0 \text{ for infinitely many } i\}$.

3. **A homeomorphism of l_2 onto s.** Throughout this section we regard $s = \prod_{i>0} I_i^0$ and let $C_i = \{y \in s \mid \sum y_j^2 \leq i \text{ and for each } j > 0, |y_j| \leq 1 - 1/i\}$. Note that C_i is compact since it is a closed subset of the product of closed intervals.

The main result of the paper is the following.

THEOREM 3.1. $l_2 \sim s$.

The plan for showing that $l_2 \sim s$ is to exhibit homeomorphisms through middle spaces as follows:

$$l_2 \sim l_2 \setminus \bigcup E^i \sim S_1 \cap (l_2 \setminus \bigcup E^i) \sim s \setminus \bigcup C_i \sim s.$$

The four homeomorphisms are guaranteed by Corollary 9.3, Lemma 3.3, Lemma 3.2, and Corollary 9.4 respectively.

The topology of the first three spaces above is determined by the topology of l_2 while the topology of the last two is determined by the topology of s, so we may start by considering the bridge between l_2 and s, namely the homeomorphism of $S_1 \cap (l_2 \setminus \bigcup E^i)$ onto $s \setminus \bigcup C_i$. This bridge homeomorphism suggests why we chose to define a homeomorphism on $S_1 \cap (l_2 \setminus \bigcup E^i)$ instead of on l_2.

LEMMA 3.2. $S_1 \cap (l_2 \setminus \bigcup E^i) \sim s \setminus \bigcup C_i$.

PROOF. Throughout this proof we denote $S_1 \cap (l_2 \setminus \bigcup E^i)$ by S_1^*. Hence the coordinates of points of both S_1^* and s lie in $(-1, 1)$.

There are many 1-1 functions taking S_1^* onto a dense subset of s. Since most of the coordinates of each point of S_1^* are close to 0, one might hope to divide each of these coordinates by a suitable divisor and obtain the corresponding coordinate of the corresponding point of s. Let us consider such an elementary function h: $S_1^* \to s$ where we let

$$x = (x_1, x_2, \cdots) \in S_1^*, \quad \text{and} \quad h(x) = (y_1, y_2, \cdots) \in s.$$

We note that $x_1 \in (-1, 1)$ and $y_1 \in (-1, 1)$ so we set $y_1 = x_1$.

Once x_1 is fixed, x_2 has the limited domain $(-(1-x_1^2)^{1/2}, (1-x_1^2)^{1/2})$ while y_2 can still be anywhere in $(-1, 1)$. We take the domain of x_2 linearly onto the domain of y_2 and thus define

$$y_2 = x_2/(1 - x_1^2)^{1/2}.$$

Continuing in this fashion, we define

$$y_i = x_i \Big/ \left(1 - \sum_1^{i-1} x_j^2\right)^{1/2}.$$

One may note that the function h we have defined is 1-1 and coordinate wise continuous. Since s is a product space, coordinatewise continuity into s implies continuity. Hence h is continuous and thus is a map.

One reason for working with S_1^* rather than S_1 is that we would have had difficulties trying to avoid dividing by 0.

In order to study h^{-1}, we may solve for $(x_1, x_2, \cdots)$ in terms of $(y_1, y_2, \cdots)$. One finds that $h^{-1} = g | h(S_1^*)$ where g: $s \to l_2$ is defined by

$$x_i = y_i(1 - y_1^2)^{1/2}(1 - y_2^2)^{1/2} \cdots (1 - y_{i-1}^2)^{1/2}.$$

Also, g is 1-1 and coordinatewise continuous. Coordinatewise convergence in l_2 does not imply convergence as can be seen by considering the sequence of points such that the ith member of the sequence has all coordinates 0 except the ith which is 1/2. However, it is well known and easy to prove that coordinatewise continuity into S_1 does imply continuity. Therefore $h^{-1} = g | h(S_1^*)$ is continuous and h is a homeomorphism. (Incidentally, it may be shown that g defined on all of s is not a homeomorphism.) Another reason for defining the bridge homeomorphism on S_1^* instead of on l_2 is that functions into S_1 are continuous if they are coordinatewise continuous.

The remaining part of the proof of Lemma 3.2 is to show that $s \setminus h(S_1^*) = \bigcup C_i$. This equality is no accident since the definition of C_i was chosen only after the definition of h.

One finds from inductive use of the equation describing the x_i's in terms of the y_i's that

$$\sum_1^i x_j^2 = 1 - (1 - y_1^2)(1 - y_2^2) \cdots (1 - y_i^2).$$

Hence, if $y \in s$, the sum of the squares of the first i coordinates of $g(y)$ is less than 1. In fact, one finds that

$$g(s) = \left\{ x \in l_2 \,\middle|\, \text{for each } i, \sum_1^i x_j^2 < 1 \right\}$$

since for each $x \in l_2$ such that for each i, $\sum_1^i x_j^2 < 1$, one can inductively define y_i coordinates so that $g(y) = x$. In particular, $S_1^* \subset g(s)$ and

$$s \setminus h(S_1^*) = g^{-1} \left\{ x \in l_2 \,\middle|\, \sum_1^\infty x_i^2 < 1 \right\}.$$

To prove that $h(S_1^*) = s \setminus \bigcup C_i$, we need only show that if $y \in h(S_1^*)$, $\sum_1^\infty y_i^2 = \infty$ and if $y \in s \setminus h(S_1^*)$, $\sum_1^\infty y_i^2$ is finite.

If $y \in s \setminus h(S_1^*)$ and $g(y) = x$, then $\sum_1^\infty x_i^2 < 1$. Since

$$y_i = x_i \Big/ \left(1 - \sum_1^{i-1} x_j^2\right)^{1/2}, \quad \text{then } y_i^2 \leqq x_i^2 \Big/ \left(1 - \sum_1^\infty x_j^2\right)$$

$$\text{and } \sum y_i^2 \leqq \sum x_i^2 \Big/ \left(1 - \sum_1^\infty x_j^2\right).$$

Hence, $\sum_1^\infty y_i^2$ is finite.

If $y \in h(S_1^*)$, we use a plan suggested by Robert Connelly and show that $\sum_1^\infty y_i^2 = \infty$ by demonstrating that for each integer n, $\sum_n^\infty y_i^2 > 1$. This is because

$$\sum_n^\infty y_i^2 = \frac{x_n^2}{\sum_n^\infty x_j^2} + \frac{x_{n+1}^2}{\sum_{n+1}^\infty x_j^2} + \cdots > \frac{x_n^2 + x_{n+1}^2 + \cdots}{\sum_n^\infty x_j^2} = 1.$$

Lemma 3.3. $l_2 \setminus \bigcup E^i \sim S_1 \cap (l_2 \setminus \bigcup E^i)$.

Proof. The required homeomorphism is $h_1 \circ h_2$ where h_2 is the homeomorphism which takes each point $(x_1, x_2, \cdots)$ to $(1, x_1, x_2, \cdots)$ and for each $x \in h_2(l_2)$, $h_1(x)$ is the point of S_1 between x and $(-1, 0, 0, \cdots)$.

4. **Convergence of homeomorphisms.** It is trivial to verify that the product (or composition) of any two homeomorphisms of a space X onto itself is a homeomorphism of X onto itself. Therefore, any finite product of such homeomorphisms is a homeomorphism of X onto itself. However, infinite left products, that is products of the form $\cdots \circ h_i \circ \cdots \circ h_2 \circ h_1$, may or may not converge so as to define a homeomorphism of X onto itself or even of a subset of X onto X.

Equivalently, we may consider the possible convergence of the sequence $h_1, h_2 \circ h_1, \cdots$ of homeomorphisms of X onto itself. We may denote this sequence by $(h_i \circ \cdots \circ h_2 \circ h_1)_{i>0}$. Frequently h_i is close to the identity so that members of the sequence are products of the previous element of the sequence by a homeomorphism near the identity.

There are two basic types of procedures which we shall use to establish that sequences of the form $(g_i)_{i>0} = (h_i \circ \cdots \circ h_2 \circ h_1)_{i>0}$ converge to desired homeomorphisms of some subset Y of X onto X.

Convergence Procedure I. A simple form of this procedure is represented by the following trivial but useful theorem.

THEOREM 4.1. *The sequence $(g_i)_{i>0}$ of homeomorphisms of X onto X converges to a homeomorphism of the subset Y of X onto X provided that*

(1) *for each $p \in Y$ there exists a neighborhood U of p in X and an integer $n(U)$ such that for each $n > n(U)$, $g_n | U = g_{n(U)} | U$ and*

(2) *for each $q \in X$ there exists a neighborhood V of q and an integer $n(V)$ such that for each $n > n(V)$, $g_n^{-1} | V = g_{n(V)}^{-1} | V$ and $g_n^{-1}(V) \subset Y$.*

Observe that if for each $i > 0$, $g_i = h_i \circ \cdots \circ h_2 \circ h_1$, then the condition that $g_n | U = g_{n(U)} | U$ becomes the condition that h_n restricted to $h_{n(U)} \circ \cdots \circ h_2 \circ h_1(U)$ is the identity restricted to $h_{n(U)} \circ \cdots \circ h_2 \circ h_1(U)$. Similarly, the condition $g_n^{-1} | V = g_{n(V)}^{-1} | V$ becomes $h_n^{-1} | V = \mathrm{Id} | V$.

Convergence Procedure II. In some instances it does not seem feasible to construct a desired homeomorphism by a sequential process using Convergence Procedure I. To handle such cases we introduce another procedure. While a simpler version of this convergence procedure has been applied to compact spaces, as far as the authors know, the formulation we give for complete metric spaces is new.

If G is an open covering of a space X and h is a homeomorphism of X, we use $h(G)$ to denote the collection of images of elements of G. Suppose $(h_i)_{i>0}$ is a sequence of homeomorphisms of a complete metric space X onto itself. We say that $(h_i)_{i>0}$ satisfies the *inductive convergence criterion* if there is a sequence $(G_i)_{i>0}$ of open coverings of X such that for each positive integer i,

mesh $G_i < 1/2^i$,
$\operatorname{mesh}(h_i \circ \cdots \circ h_2 \circ h_1)^{-1}(G_i) < 1/2^i$, and
h_{i+1} is limited by G_i.

THEOREM 4.2. *If $(h_i)_{i>0}$ is a sequence of homeomorphisms of a complete metric space X onto itself which satisfies the inductive convergence criterion, then $(g_i)_{i>0} = (h_i \circ \cdots \circ h_2 \circ h_1)_{i>0}$ converges to a homeomorphism of X onto itself.*

PROOF. For each $p \in X$, $(g_i(p))_{i>0}$ is a Cauchy sequence since $g_i(p)$ and $h_{i+1} \circ g_i(p) = g_{i+1}(p)$ lie in some one element of G_i and mesh $G_i < 1/2^i$. This implies that the limit $(g_i)_{i>0}$ is uniquely defined. Since each g is continuous and $d(g_i(p), g_{i+1}(p)) < 1/2^i$ for all p, the limit $(g_i)_{i>0}$ is continuous.

Also, for any $q \in X$, q and $h_{i+1}^{-1}(q)$ lie in some one element of G_i. Since mesh $g_i^{-1}(G_i) < 1/2^i$, the distance between $g_i^{-1}(q)$ and $g_i^{-1} \circ h_{i+1}^{-1}(q) = g_{i+1}^{-1}(q)$ is less than $1/2^i$. Hence $(g_i^{-1}(q))_{i>0}$ converges. Indeed $(g_i^{-1})_{i>0}$ converges to a continuous function and thus $(g_i)_{i>0}$ converges to a homeomorphism of X onto itself.

The following is a variation of Theorem 4.2 that we shall use. If A, B are sets we use $d(A, B)$ to denote the greatest lower bound of $d(a, b)$ where $a \in A$, $b \in B$.

THEOREM 4.3. *Suppose $(K_i)_{i>0}$ is a sequence of closed sets in a complete metric space X and for each positive integer i, h_i is a homeomorphism of $X \backslash h_{i-1} \circ \cdots \circ h_2 \circ h_1 \circ \operatorname{Id}(K_i \backslash \bigcup_1^{i-1} K_j)$ onto X. Let G_i be an open covering of $X \backslash \bigcup_1^{i-1} K_j$ such that* mesh $G_i < 1/2^i$, mesh $h_i \circ \cdots \circ h_2 \circ h_1(G_i) < 1/2^i$, *and if $g \in G_i$,* diameter $g < d(g, \bigcup_1^{i-1} K_j)/2^i$. *Then $(h_i \circ \cdots \circ h_2 \circ h_1)_{i>0}$ converges to a homeomorphism of $X \backslash \bigcup K_i$ onto X if each h_{i+1} is limited by $h_i \circ \cdots \circ h_2 \circ h_1(G_i)$.*

The proof is similar to the proof of Theorem 4.2. The condition that diameter $g < d(g, \bigcup_1^{i-1} K_j)/2^i$ is used to show that $(h_1^{-1} \circ h_2^{-1} \circ \cdots \circ h_i^{-1}(q))_{i>0}$ converges to a point of $X \backslash \bigcup K_j$ rather than to a point of $\bigcup K_j$.

5. Isotopies. Several times in this paper we shall be concerned with isotopies, which we define for our purposes as follows. For a metric space X, an *isotopy* H of X onto X is a continuous 1-parameter family of homeomorphisms H_t $(0 \leqq t \leqq 1)$ of X onto X such that $H_0 = \operatorname{Id}$; it is required that H be simultaneously continuous in t and X rather than that homeomorphisms be near each other.

One can think of an isotopy as a motion of X onto itself starting with the identity H_0, ending with H_1, and using t as a time variable.

Although it is required that an isotopy H be continuous in both t and X, it is not ordinarily required that $\{H_t^{-1}\}_{0\leq t\leq 1}$ be continuous in this sense. If it is, we call the isotopy *invertible*. In this paper we shall only use isotopies that are invertible.

For compact spaces, all isotopies are invertible, but this is not true for spaces in general. A noninvertible isotopy H of l_2 onto itself can be defined as follows. Let f_i be a map of $[0, 1]$ onto $[1/i, 1]$ such that $f_i[0, 1/i+1]=f_i[1/i, 1]=1$. For each $(x_1, x_2, \cdots)\in l_2$, let $H_t(x_1, x_2, \cdots)=(x_1\cdot f_1(t), x_2\cdot f_2(t), \cdots)$. Although H_0^{-1} is continuous in l_2 and $H_t^{-1}(0, 0, \cdots)$ is continuous in t, H^{-1} is not continuous in t and l_2 at $t=0$, $(0, 0, \cdots)\in l_2$.

For $K\subset X$, an *invertible isotopy H pushing K off X is a* 1-parameter family of homeomorphisms H_t $(0\leq t\leq 1)$ onto X such that

$$\begin{aligned}&H_0=\mathrm{Id},\\ &H_t(X)=X \quad \text{for } (0\leq t<1),\\ &H_1(X\backslash K)=X,\end{aligned}$$

and

$$H \text{ and } H^{-1} \text{ are continuous in } t \text{ and } X.$$

Those who prefer to think of an isotopy as a map H of $X\times[0, 1]$ onto X that is an onto homeomorphism at every level may regard this invertible isotopy pushing K off X as a map from $(X\times[0,1])\backslash(K\times\{1\})$ onto X satisfying the appropriate conditions.

If K is a single point p, we say that H is an *invertible isotopy pushing p off X.*

In Lemma 7.1 and Step 1 of Lemma 8.2 we exhibit invertible isotopies pushing points off s and a copy $l_2(1)$ of l_2. The motions in the two cases are rather different. The first pushes a point off s in some one direction while the second pushes a point off $l_2(1)$ by increasing its norm (distance from the origin) using infinitely many directions. In both instances, the isotopies pushing individual points off the spaces are the key devices in pushing certain countable unions of closed sets of infinite deficiency off s and $l_2(1)$.

For any isotopy H on a space X and numbers a, b $(0\leq a<b\leq 1)$, we let $H[a, b]$ denote the isotopy which acts on the interval $[a, b]$ as H acts on $[0, 1]$. Specifically,

$$H[a, b]_t = \begin{cases}\mathrm{Id} & \text{for } 0 \leq t \leq a,\\ H_1 & \text{for } b \leq t \leq 1,\\ H_{(t-a)/(b-a)} & \text{for } a \leq t \leq b.\end{cases}$$

Observe that for H and F isotopies on X, a motion which is essentially H followed by F may be represented as the isotopy defined for each t $(0 \leqq t \leqq 1)$ and any a, b, c $(0 \leqq a < b < c \leqq 1)$ as $F[b, c]_t \circ H[a, b]_t$.

The following proposition is a variation of Theorem 4.1.

LEMMA 5.1. *Suppose that for each positive integer i, H^i is an invertible isotopy of X onto X and $p_0 \in X$. Then*

$$H = \left\{ L \prod_{i>0} H^i[1 - (1/i), 1 - (1/i + 1)]_t \right\}$$

is an invertible isotopy pushing p_0 off X provided that

(1) *for each $p \in X \setminus \{p_0\}$, there is a neighborhood U of p in X and an integer $n(U)$ such that for each $n > n(U)$, $H_t^n = \mathrm{Id}$ on $H_1^{n(U)} \circ \cdots \circ H_1^2 \circ H_1^1(U)$ and*

(2) *for each $q \in X$, there is a neighborhood V of q in X and an integer $n(V)$ such that for each $n > n(V)$, $(H_t^n)^{-1} = \mathrm{Id}$ on V and $(H_1^{n(V)} \circ \cdots \circ H_1^2 \circ H_1^1)^{-1}(V) \subset X \setminus \{p_0\}$.*

PROOF. For any t $(0 \leqq t < 1)$, let i be such that $1 - (1/i) > t$. Then $H^i[1 - (1/i), 1 - (1/i + 1)]_t$ is the identity. Hence H_t may be considered as a finite product of homeomorphisms of X onto itself. Hence, the continuity of H and H^{-1} for values of t in $(0 \leqq t < 1)$ follows from the continuity of the finite factors to be considered.

For $x_0 \in X \setminus \{p_0\}$ and $t = 1$, the continuity of H at $(x_0, 1)$ follows from Condition 1 in the statement of the lemma while if $x \in X$, the continuity of H^{-1} at $(x, 1)$ follows from Condition 2.

6. Straightening sets in s.

There are two related ways of showing that if a countable collection of compact sets are pushed out of s, the remainder is homeomorphic with s. In one of these procedures, s is regarded as the pseudo interior of the Hilbert cube. This approach was used in [2]. The procedure used in [2] offers an alternative (and perhaps easier) method of proving Corollary 9.4. However, in the present paper we adopt a procedure that parallels our treatment of the proof that $l_2 \sim l_2 \setminus \cup E^i$. In §§ 6 and 7 we regard s as $\prod_{i>0} R_i$ where R_i denotes a copy of the real line. In this section we show how to move sets into such nice positions that they can be pushed out of s with a procedure to be described in §9.

LEMMA 6.1. *For any compact subset C of s there exists a homeomorphism h of s onto s such that $h(C)$ projects onto a single point of R_1—that is, $\tau_1 h(C)$ is a single element.*

PROOF. The proof is in two steps. In the first step we adjust C so that its image intersects any line parallel to the R_1-axis in at most one point. In the second step we move points in the 1st direction so that the final image of C lies in a hyperspace of s perpendicular to the R_1-axis.

Step 1. For each $i>1$, let f_i be a homeomorphism of $R_1\times R_i$ onto itself such that (1) f_i does not change the 1st coordinate of any point and (2) for some closed rectangular region D_i in $R_1\times R_i$ with $\tau_{\{1,i\}}(C)\subset D_i$, $f_i(D_i)$ is a closed parallelogram region intersecting no line parallel to the R_1-axis in a set of diameter greater than $1/2^i$.

Let f be the homeomorphism of s onto itself such that $f(x_1, x_2, \cdots) = (x_1, y_2, y_3, \cdots)$ where y_i is the 2nd coordinate of $f_i(x_1, x_i)$. Each point of s has a unique image in s and a unique inverse image in s determined coordinatewise. Continuity follows from the continuity of coordinate functions determined by the f_i's.

For any $i>0$, we know from the definition of f_i that $f(C)$ cannot intersect any line parallel to the R_1-axis in a set of diameter more than $1/2^i$. Therefore $f(C)$ intersects any such line in at most a single point.

Step 2. Let $s(0)$ be the set of points in s whose 1st coordinate is 0 and τ^* denote the projection function of s onto $s(0)$. Then $\tau^*|f(C)$ is a homeomorphism into $s(0)$.

Let $\phi:\tau^*(f(C))\rightarrow R_1$ be defined by $\phi=\tau_1\tau^{*-1}|\tau^*(f(C))$. By the Tietze Extension Theorem there is a map $\Phi:s(0)\rightarrow R_1$ such that $\Phi|\tau^*(f(C))=\phi$.

Let h^* be the homeomorphism of s onto itself such that for each point $p\in s(0)$ and line L_p through p parallel to the R_1-axis, $h^*|L_p$ is a translation of $-\Phi(p)$ units in the 1st direction. The homeomorphism promised by Lemma 6.1 is $h=h^*\circ f$. Note that $h(C)\subset s(0)$ and thus $\tau_1(h(C))=0$.

THEOREM 6.2. *For each collection* $\{C_i\}_{i>0}$ *of compact subsets of* s *there is a homeomorphism* g *of* s *onto* s *such that each* $g(C_i)$ *is infinitely deficient.*

PROOF. Let $\alpha_1, \alpha_2, \cdots$ be disjoint infinite subsets of Z such that $\bigcup\alpha_i=Z$. Let s be written as $\prod_{i>0} s^{\alpha_i}$ where $s^{\alpha_i}=\prod_{j\in\alpha_i} R_j$.

Let $\theta:Z\rightarrow Z$ be such that for each $i>0$, $\theta^{-1}(i)$ is infinite. We regard each s^{α_i} as a copy of s and learn from Lemma 6.1 that there is a homeomorphism g_i of s^{α_i} onto itself such that $g_i(\tau_{\alpha_i}(C_{\theta(i)}))$ is deficient with respect to the first element of α_i.

Let g be the homeomorphism of $s=\prod_{i>0} s^{\alpha_i}$ onto itself defined coordinatewise as g_i on s^{α_i}. From the definitions of θ and of the g_i's it follows that for each $i>0$, $g(C_i)$ is infinitely deficient.

7. Isotopies pushing points off s. We shall need to consider a 2-parameter ($0<r\leqq 1$, $0\leqq t\leqq 1$) family of homeomorphisms ${}_rH_t$ such that for r fixed, ${}_rH$ is an isotopy pushing the origin off s. We call such a family an *invertibly continuous family of invertible isotopies* if both ${}_rH_t$ and ${}_rH_t^{-1}$ are continuous in r, t, and s. We shall want each ${}_rH_t$ to be the identity outside the r-neighborhood of the origin in s. We are supposing that s has the complete metric defined by $d(x, y) = \sum(\min(1/2^i, |x_i-y_i|))$.

We first describe an invertible isotopy pushing the origin off s and then modify the isotopy to obtain invertibly continuous families of invertible isotopies each pushing the origin off s.

LEMMA 7.1. *There is an invertible isotopy H pushing the origin p_0 off s.*

PROOF. We shall push p_0 off s along the positive ray of the x_1-axis. For convenience, we regard each finite product of lines as having a Euclidean metric.

For each positive integer i, let F^i be an invertible isotopy on $R_1\times R_{i+1}$ such that (1) F_t^i leaves each point whose 1st coordinate is less than $i-1-1/2^{i-1}$ fixed while (2) F_1^i carries the vertical ray from $(i-1, -1/2^i)$ through $(i-1, 1)$ isometrically onto the horizontal ray from $(i-1/2^i, 0)$ through $(i+1, 0)$. The isotopy F^i can result from a "limited rotation" about $(i-1, 0)$ (not all points of the plane being rotated) followed by a "limited translation."

Let $a\in R_2\times\cdots\times R_i$, $b\in R_1\times R_{i+1}$, and $c\in\prod_{j>i+1} R_j$. For any $i>1$ and any $a\in R_2\times\cdots\times R_i$ let $\phi_i(a)=\max\{0, 1-2^{i+1}\cdot d(0, a)\}$ where $d(0, a)$ denotes the distance from a to the origin of Euclidean $(i-1)$-space $R_2\times\cdots\times R_i$. To avoid a special argument, we suppose that ϕ_1 is the constant 1. Let H^i be an isotopy on s defined for each t ($0\leqq t\leqq 1$) as

$$H_t^i(a, b, c) = (a, F^i_{t\cdot\phi_i(a)}(b), c).$$

When we verify that $H^1, H^2, \cdots$ satisfies Conditions 1 and 2 of the statement of Lemma 5.1, it will follow from that lemma that

$$H = \left\{L\prod_{i>0} H^i[1-1/i, 1-1/i+1]_t\right\}$$

is an invertible isotopy pushing p_0 off s. Let h^i denote $H_1^i\circ\cdots\circ H_1^2\circ H_1^1$.

(1) To see that the H^i's satisfy Condition 1, one need only consider the geometry of the successive motions. Specifically, we let $p\in s\backslash p_0$ with $p=(0, \cdots, a_j, a_{j+1}, \cdots)$ where a_j is the first nonzero coordinate of p. Let $h^j(p)=(b_1, b_2, \cdots, b_j, b_{j+1}, a_{j+2}, \cdots)$ and $h^{j+1}(p)$

$=(c_1, b_2, \cdots, b_{j+1}, c_{j+2}, a_{j+3}, \cdots)$. If $j>1$, each of $b_2, \cdots, b_j$ is zero but we shall not use this information. If any one of $b_2, \cdots, b_{j+1}, c_{j+2}$ is nonzero, there is a neighborhood U of p and an integer k such that $\phi_k=0$ on the projection of $h^k(U)$ into $R_2\times \cdots \times R_k$. Hence for $k'\geqq k$, $H_t^{k'}=\mathrm{Id}$ on $h^k(U)$. If each of $b_2, \cdots, b_{j+1}$ is zero, then $b_1\neq j$ since $h^j(0, \cdots, 0, a_{j+2}, \cdots)=(j, 0, \cdots, 0, a_{j+2}, \cdots)$. If $b_1\neq j$ and $c_{j+2}=0$, $c_1<j+1-1/2^{j+1}$ since F_1^{j+1} takes the ray from $(j, -1/2^{j+1})$ through $(j, 1)$ onto the ray from $(j+1-1/2^{j+1}, 0)$ through $(j+2, 0)$. But if $c_1<j+1-1/2^{j+1}$, there is a neighborhood U of p such that the first coordinate of each point of $h^{j+1}(U)$ is less than $j+1-1/2^{j+1}$. Then for $k\geqq j+1$, $H_t^k=\mathrm{Id}$ on $h^{j+1}(U)$.

(2) To see that the H^i's satisfy Condition 2, it is only necessary to notice that H_t^{i+1} does not move a point unless its first coordinate is greater than $i-1/2^i$ and the first coordinate of $h^i(p_0)$ is i.

The following lemma is included as a stepping stone toward Lemma 7.3.

LEMMA 7.2. *Suppose s is expressed as $s=s^\alpha\times s^{\alpha'}$ as suggested in Lemma 2.1 where s^α, $s^{\alpha'}$ are copies of s with origins p_0^α, $p_0^{\alpha'}$. Then if H is an invertible isotopy pushing p_0^α off s^α and ϕ_r is a continuous 1-parameter $(0<r\leqq 1)$ family of maps of $s^{\alpha'}$ into $[0, 1]$ such that for each r, $\phi_r^{-1}(1)=p_0^{\alpha'}$, then*

$$ {}_rH_t(p, q) = (H_{t\cdot\phi_r(q)}(p), q) $$

defines an invertibly continuous 1-parameter $(0<r\leqq 1)$ family of invertible isotopies each pushing the origin off s.

The proof is left to the reader.

LEMMA 7.3. *There is an invertibly continuous 1-parameter $(0<r\leqq 1)$ family of invertible isotopies ${}_rH$ each pushing the origin p_0 off s such that ${}_rH_t$ is the identity outside the r-neighborhood of p_0.*

PROOF. For any r $(0<r\leqq 1)$, ${}_rH$ is to be an isotopy whose "action" pushes the origin off a certain subspace s_0 of s and is "phased out" in a neighborhood U of the origin in the complementary subspace s_0' (i.e., $s=s_0\times s_0'$). The two "factors" which determine the size of the domain of support of the isotopy ${}_rH$ are (1) the diameter of the subspace s_0 and (2) the diameter of the neighborhood U of the origin in s_0'. We shrink the former, as r tends to zero, by a process of swapping axes and the latter by a reduction process introduced after each swap of axes.

Let α be an infinite subset of Z such that α' is infinite and $1,2,3\in\alpha'$. The distances in s^α and $s^{\alpha'}$ are given by $d(x, y)$

$=\sum_{i\in\alpha}(\min(1/2^i, |x_i-y_i|))$ and $d(x, y)=\sum_{j\in\alpha'}(\min(1/2^j, |x_j-y_j|))$ respectively. Note that the diameter of s^α is less than $1/2^3$.

Let H be an isotopy pushing the origin off s^α as guaranteed by Lemma 7.1.

Let ϕ be a map of $s^{\alpha'}$ onto $[0, 1]$ such that $\phi^{-1}(1)=p_0^{\alpha'}$ and $\phi=0$ for all points outside a 1/8-neighborhood of $p_0^{\alpha'}$. We could have specified that $\phi(q)=\max[0, \ 1-8\sum_{i\in\alpha'}(\min \ 1/2^i, \ |x_i|)]$ for $q=(x_{n_1}, x_{n_2}, \cdots)$ but we shall not be concerned with the particular description of ϕ. In fact, it is perhaps best not to define ϕ in terms of the distance from $p_0^{\alpha'}$ since this distance changes when coordinates are interchanged. We define ${}_1H$ so that for each $(p, q)\in s^\alpha\times s^{\alpha'}$ (with $(p, q)\neq p_0$ for $t=1$)

$${}_1H_t(p, q) = (H_{t\cdot\phi(q)}(p), q).$$

Note that ${}_1H_t(p, q)=(p, q)$ if $d((p, q), p_0)\geqq 1/4$. This is because $1, 2, 3\in\alpha'$ and $d((p, q), p_0)\geqq 1/4$ implies $d(q, p_0^{\alpha'})\geqq 1/8$ and $\phi(q)=0$.

As r changes from 1 to 1/2, we define ${}_rH$ by modifying ${}_1H$ by swapping axes and reducing ϕ. We describe each of these operations separately.

Swapping axes. Our purpose of swapping axes is to move the small positive integers out of α and hence reduce the diameter of s^α. We recall that 1, 2, 3 are in α'. Let j be the least integer in α and k be an integer in α' larger than j. Let α_1' be the subset of Z obtained from α' by replacing k by j and α_1 be the subset obtained from α by replacing j by k.

The action of the isotopy ${}_1H$ is defined on s^α and is phased out in a neighborhood of the origin in $s^{\alpha'}$. We gradually transfer the action over $3/4\leqq r\leqq 1$ so that the action of ${}_{3/4}H$ will be defined on s^{α_1} and be phased out in a neighborhood of the origin in $s^{\alpha_1'}$. The following formulas describe a suitable version of this process.

For $0\leqq\lambda\leqq 1$, let F_λ be the rotation of s such that if $F_\lambda(x_1, x_2, \cdots)=(y_1, y_2, \cdots)$, then $y_i=x_i$ unless $i\in\{j, k\}$ and (y_j, y_k) is the image of (x_j, x_k) under a clockwise rotation of the plane $R_j\times R_k$ by $\lambda\cdot\pi/2$ about the origin. Let $f(r)$ be the linear function that sends 1 to 0 and 3/4 to 1. Then for $3/4\leqq r\leqq 1$,

$${}_rH_t = F_{f(r)}^{-1}\circ {}_1H_t\circ F_{f(r)}.$$

We note that if $d(x, p_0)\geqq 1/2$, then ${}_rH_t(x)=x$ if $3/4\leqq r\leqq 1$. This is because for such x $d(F_{f(r)}(x), p_0)\geqq 1/2-1/2^j\geqq 1/4$, ${}_1H_t\circ F_{f(r)}(x)=F_{f(r)}(x)$ and $F_{f(r)}^{-1}\circ {}_1H_t\circ F_{f(r)}(x)=x$.

Reducing ϕ. Here we define ${}_rH$ for $1/2\leqq r\leqq 3/4$. Let $x=(p', q')$

where $p' \in s^{\alpha_1}$, $q' \in s^{\alpha_1'}$. From the definitions of ${}_1H$ and F_λ and specifically from the fact that the rotation of $R_j \times R_k$ which determines F_1 merely interchanged vertical and horizontal lines in $R_j \times R_k$, it follows that

$$_{3/4}H_t(p', q') = (H_t'\cdot_{\phi_{3/4}(q')}(p'), q')$$

where $\phi(3/4)$ is defined below and where H_t' is $T^{-1} \circ H_t \circ T$ applied to s^{α_1} instead of s^α with T being a permutation of a finite number of coordinates. The reason for introducing T is that k may not be the least integer in α_1 and in this case we need to shift k to the first element of α_1 and to move all the others in α_1 which are less than k into their successors in α_1. Also, $\phi_{3/4}$ is a map of $s^{\alpha_1'}$ onto $[0, 1]$ such that $\phi_{3/4}^{-1}(1) = p_{01}^{\alpha'}$ and $\phi_{3/4} = 0$ except in a small neighborhood of the origin $p_{01}^{\alpha}{}'$. In fact, $\phi_{3/4} = \phi \circ T'$ where T' is the map from $s^{\alpha_1'}$ onto $s^{\alpha'}$ which merely moves the jth coordinate of a point to the kth coordinate and changes its sign. Note that if $d(x, p_0) \geq 1/2$, then $\phi_{3/4}(q') = 0$ as $d(q', p_{01}^{\alpha'}) \geq 1/2 - 1/16$ and $d(T'(q'), p_0^{\alpha'}) \geq 1/2 - 1/16 - 1/32 \geq 1/8$ which implies that $\phi_{3/4}(q') = \phi(T'(q')) = 0$.

Let $\phi_{1/2}$ be a map of $s^{\alpha_1'}$ onto $[0, 1]$ so that $\phi_{1/2} \leq \phi_{3/4}$, $\phi_{1/2}^{-1}(1) = p_{01}^{\alpha'}$, and $\phi_{1/2} = 0$ except in the $1/2^4$-neighborhood of $p_{01}^{\alpha'}$. For each $q' \in s^{\alpha_1'}$ let, $\phi_r(q')$ divide $\phi_{1/2}(q')$, $\phi_{3/4}(q')$ in the same ratio that r divides $1/2$, $3/4$ with $\phi_r(q') = \phi_{1/2}(q')$ if $\phi_{1/2}(q') = \phi_{3/4}(q')$. Then for $1/2 \leq r \leq 3/4$, define

$$_rH_t(p', q') = (H'_{t\cdot\phi_r(q')}(p'), q').$$

As r moved from 1 to $1/2$, we defined ${}_rH_t$ from ${}_1H_t$ by swapping axes and reducing ϕ. By similarly swapping axes so as to replace the least element of α_1 with a larger element of α_1' and by further reducing ϕ, we define ${}_rH_t$ $(1/4 \leq r \leq 1/2)$ from ${}_{1/2}H_t$ in an analogous fashion. Similarly, we define ${}_rH_t$ as r shrinks to $1/2^3$, to $1/2^4$, $\cdots$.

8. **Isotopies pushing points off l_2.** We prove lemmas about pushing points off l_2 similar to those we proved about pushing points out of s in the last section. We recall that S_1 is the set of all points of l_2 at a distance of 1 from the origin.

LEMMA 8.1. *There is an invertible isotopy F pushing a point off S_1.*

PROOF. Let $l_2(1)$ be the set of all points of l_2 whose first coordinate is 1. For each positive integer i, let p_i denote the point of l_2 whose first i coordinates are each 1 and whose other coordinates are 0. We prove Lemma 8.1 in two steps.

Step 1. In this step we exhibit a particular invertible isotopy H

on $l_2(1)$ pushing p_1 out of $l_2(1)$. The plan is to push p_1 to p_2, then to p_3, then to $p_4, \cdots$, and hence off l_2. We take care that no point other than p_1 undergoes more than a finite number of these pushes.

For each integer $i \geqq 2$, let U_i be the neighborhood of p_i in $l_2(1)$ such that

$$U_i = \left\{(1, x_2, \cdots) \in l_2(1) \mid \sum_2^i (1 - x_j)^2 < 1/i^2\right\}.$$

Let H^1 be the isotopy on $l_2(1)$ such that H_t^1 is a translation of t units in the x_2 direction. For each integer $i \geqq 2$, H^i is the isotopy on $l_2(1)$ such that $H_t^i(p)$ is the point resulting from translating p by $i \cdot t \cdot d(p, (l_2(1) \setminus U_i))$ in the x_{i+1} direction. Then

$$H_t = L \prod_{i>0} H^i[1 - 1/i, 1 - 1/(i+1)]_t.$$

It may be shown with the use of Theorem 4.1 and Lemma 5.1 that H is an invertible isotopy pushing p_1 out of $l_2(1)$.

Step 2. In this step we describe an invertible isotopy F pushing p_1 off S_1. Let S_1^+ be the set of all points of S_1 with positive 1st coordinate and ρ be the projection of S_1^+ onto $l_2(1)$ from the origin. The invertible isotopy F promised by Lemma 8.1 may be defined as

$$\begin{aligned} F_t(p) &= \rho^{-1} \circ H_t \circ \rho(p) \quad \text{if } p \in S_1^+, \\ &= p \quad \text{if } p \in S_1 \setminus S_1^+. \end{aligned}$$

To complete the proof we need only verify the continuity of F and F^{-1} at points $p \in S_1$ whose 1st coordinate is 0. Suppose the ith coordinate of p is not 0 and δ is one third the absolute value of this coordinate. A calculation shows that if U is the δ-neighborhood of p in S_1 and $q \in \rho(U \cap S_1^+)$, then the absolute value of the ith coordinate of q is greater than 2. For $k \geqq i$, H_t^k and $(H_t^k)^{-1}$ do not move q. Hence, to verify the continuity of F and F^{-1}, we only need to check the effect of a finite number of $\rho^{-1} \circ H_t^i \circ \rho$'s near p; in fact we only need to check one of them since the composition of a finite collection of continuous functions is continuous.

A computation shows that if $p = (0, x_2, x_3, \cdots) \in S_1$ and $q = (y_1, y_2, \cdots) \in S_1^+$ within δ of p, then for some $0 \leqq \theta \leqq 1$,

$$\rho^{-1} \circ (H_t^{j-1})^{-1} \circ \rho(q) = (k \cdot y_1, k \cdot y_2, \cdots, k(y_j + \theta \cdot y_1), \cdots)$$

where $k = (1 + 2 \cdot y_j \cdot y_1 \cdot \theta + y_1^2 \cdot \theta^2)^{-1/2}$. If δ is small, then y_1 is small and k is near 1. Hence coordinatewise, $\rho^{-1} \circ (H_t^j)^{-1} \circ \rho(q)$ is near p. Coordinatewise continuity in S_1 implies continuity there so F is continuous at p. Similarly, F^{-1} is continuous at p also.

LEMMA 8.2. *There is an invertible isotopy H pushing the origin p_0 off l_2 such that each H_t is the identity outside the 1-neighborhood of p_0.*

PROOF. For each point $p \neq p_0$, we let a_p denote the projection of p from p_0 on S_1.

Let ϕ be the map of the nonnegative reals onto $[0, 1]$ such that $\phi(x)=0$ if $x \geqq 1$, $\phi(0)=0$, $\phi(1/2)=1$, and ϕ is linear on each of $[0, 1/2]$ and $[1/2, 1]$.

The isotopy H promised by Lemma 8.2 is defined by $H_t = H^2[1/2, 1]_t \circ H^1[0, 1/2]_t$ where H^1 and H^2 are isotopies defined as follows. The isotopy H^1 is the identity outside the 1-neighborhood of p_0 and $H_1^1(p_0) = (1/2, 0, 0, \cdots)$. For each t $(0 \leqq t \leqq 1)$, $H_t^2(p)$ is at the same distance from p_0 as p and the projection of $H_t^2(p)$ from p_0 on S_1 is $F_{t \cdot \phi(d(p,p_0))}(a_p)$ where F is the isotopy of Lemma 8.1.

LEMMA 8.3. *There is an invertibly continuous 1-parameter $(0 < r \leqq 1)$ family of invertible isotopies ${}_rH$ each pushing the origin p_0 off l_2 such that for any t $(0 \leqq t \leqq 1)$, ${}_rH_t$ is the identity outside the r-neighborhood of p_0.*

PROOF. The required isotopy is defined by

$$ {}_rH_t = r \circ H_t \circ 1/r $$

where H is the isotopy of Lemma 8.2 and $r(p)$ is the point whose coordinates are obtained by multiplying those of p by r.

9. **Eliminating sets with small pushes.** Can a small push send a mathematician out of society without leaving a vacancy? We shall be interested in conditions under which a set can be pushed out of a space with a small push.

LEMMA 9.1. *Let X denote l_2 or s, G be an open covering of X, α be an infinite subset of Z, and K be a closed subset of X which is deficient with respect to α. Then for each open set U containing K there exists a homeomorphism h of $X \backslash K$ onto X such that h is limited by G, h is the identity except on U, and $\tau_i(x) = \tau_i(h(x))$ for each $i \in \alpha'$.*

PROOF. We suppose $\alpha' \neq \emptyset$ and write $X = X^\alpha \times X^{\alpha'}$ as suggested by Lemmas 2.1 and 2.2. With no loss of generality we suppose that $\tau_\alpha(K) = \{p_0^\alpha\}$ where p_0^α is the origin in X^α. Let $\tau_{\alpha'}(K) = K' \subset X^{\alpha'}$. See Figure 1.

We need a function r from $X^{\alpha'}$ into $(0, 1]$ to measure how far each

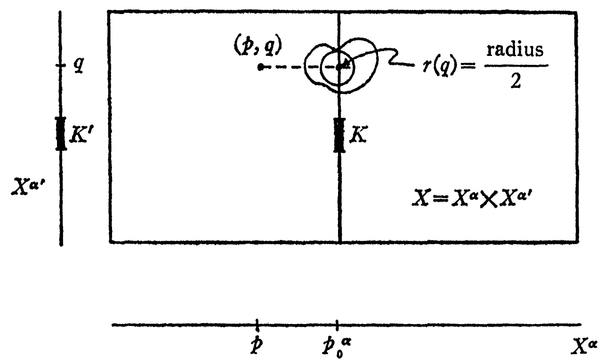

FIGURE 1

point (p_0^α, q) lies inside elements of G. Let G' be an open covering of X of mesh less than 1 such that each element of G' lies in an element of G and each element of G' intersecting K lies in U. Letting $V_\epsilon(x)$ denote the ϵ-neighborhood of x in X, we define

$$r(q) = \sup \left\{\epsilon \,\middle|\, V_{2\epsilon}(p_0^\alpha, q) \text{ lies in some element of } G'\right\}.$$

Note that for each $q \in X^{\alpha'}$, $r(q) > 0$.

Next, we need a function t from $X^{\alpha'}$ into $[0, 1]$ to measure the degree of closeness of points of $X^{\alpha'}$ to K'. Let

$$A = \left\{x \in X^{\alpha'} \,\middle|\, V_{r(x)}(p_0^\alpha, x) \not\subset U\right\}$$

and

$$t(q) = d(q, A)/(d(q, A) + d(q, K')).$$

(We supposed $A \neq \emptyset$. If $A \neq \emptyset$, define $t(q) = 1$.)

Note that $\bar{A} \cap K' = \emptyset$, $t(K') = 1$, and $t(A) = 0$.

By Lemma 8.3 for l_2 (and thus for l_2^α) and by Lemma 7.3 for s (and thus for s^α), there exists an invertibly continuous family ${}_rH$ $(0 < r \leqq 1)$ of invertible isotopies each pushing p_0^α off X^α and such that for each t $(0 \leqq t \leqq 1)$, ${}_rH_t$ is the identity outside the r-neighborhood of p_0^α in X^α.

We shall verify that the desired homeomorphism h of our lemma is

$$h(p, q) = ({}_{r(q)}H_{t(q)}(p), q).$$

Since $t(q) = 1$ for $q \in K'$, we note that for such q, ${}_{r(q)}H_{t(q)}$ is a homeomorphism of $(X^\alpha \setminus \{p_0\}) \times \{q\}$ onto $X^\alpha \times \{q\}$. Also, since $t(q) < 1$ for $q \in X^{\alpha'} \setminus K'$, we note that for such q, ${}_{r(q)}H_{t(q)}$ is a homeomorphism of

$X^\alpha \times \{q\}$ onto $X^\alpha \times \{q\}$. Since r and t are continuous functions on X and ${}_rH$ is a continuous family of isotopies, then h is continuous. Since ${}_rH$ is an invertibly continuous family of invertible isotopies and h^{-1} is described by $h^{-1}(p) = ({}_{r(q)}H^{-1}_{t(q)}(p), q)$, h^{-1} is also continuous. Hence, h is a homeomorphism of $X \backslash K$ onto X.

Since q is the second coordinate of $h(p, q)$, $\tau_i(x) = \tau_i(h(x))$ for each $i \in \alpha'$. We note that h is the identity outside U since ${}_rH_t$ is the identity outside the r neighborhood of p_0^α and $V_{r(q)}(p_0^\alpha, q) \subset U$ if $t(q) \neq 0$.

It only remains to observe that h is limited by G. Since ${}_rH_t$ is the identity except on the r-neighborhood of p_0^α in X^α, either $h(p, q) = (p, q)$ or each of p and ${}_{r(q)}H_{t(q)}(p)$ lie within $r(q)$ of p_0^α. In this latter case, the definition of $r(q)$ insures that both (p, q) and $h(p, q)$ lie in an element of G containing (p_0, q).

We are now in a position to prove a theorem having two important corollaries.

THEOREM 9.2. *Let X denote either l_2 or s, U be an open set in X, and $\{K_i\}_{i>0}$ be a countable collection of closed sets of X which lie in U, with each K_i having infinite deficiency. Then there exists a homeomorphism h of $X \backslash \bigcup K_i$ onto X such that $h | X \backslash U$ is the identity.*

PROOF. By a standard sequential process augmenting one finite set at a time, it is routine to show that there exists a collection $\{\alpha_i\}_{i>0}$ of disjoint infinite subsets of Z such that K_i is deficient with respect to α_i. Using Lemma 9.1 to assert the existence of h_i, we can inductively define a sequence of homeomorphisms $(h_i)_{i>0}$ and a sequence of coverings $(G_i)_{i>0}$ satisfying the hypotheses of Theorem 4.3 (specifying that h_i does not change the j-coordinate of any point unless $j \in \alpha_i$) so that $L \prod_{i>0} h_i$ is a homeomorphism of $X \backslash \bigcup K_i$ onto X.

COROLLARY 9.3. $l_2 \backslash \bigcup E^i \sim l_2$.

COROLLARY 9.4. *If $\{C_i\}_{i>0}$ is a countable family of compact subsets of s, then $s \backslash \bigcup C_i$ is homeomorphic to s.*

PROOF. It follows from Theorem 6.2 that there is a homeomorphism h of s onto itself such that each $h(C_i)$ is infinitely deficient. Then Corollary 9.4 follows from Theorem 9.2.

The following corollary may be useful in studying spaces which are locally like l_2 or s.

COROLLARY 9.5. *If U is an open subset of l_2 and $K_1, K_2, \cdots$ are compact subsets in U, then there is a homeomorphism h of $l_2 \backslash \bigcup K_i$ onto l_2 such that $h | l_2 \backslash U$ is the identity.*

PROOF. For convenience, we regard l_2 as s. It follows from Theorem 6.2 that there is a homeomorphism g of s onto itself such that each $g(K_i)$ is infinitely deficient. We learn from Theorem 9.2 that there is a homeomorphism h' of $s\backslash \bigcup g(K_i)$ onto s such that $h'|s\backslash g(U)$ is the identity. Then $h=g^{-1}\circ h\circ g$.

10. Summary of the procedure. In the foregoing we have established the existence of a homeomorphism between l_2 and s (Theorem 3.1). Our method of attack was to use several intermediate spaces and homeomorphisms as follows:

$$l_2 \overset{A}{\sim} l_2\backslash \bigcup E^i \overset{B}{\sim} S_1 \cap (l_2\backslash \bigcup E^i) \overset{C}{\sim} s\backslash \bigcup C_i \overset{D}{\sim} s.$$

Of the four homeomorphisms, B was easily established in Lemma 3.3 and C was established in Lemma 3.2 by exhibiting explicit formulas. The bulk of the paper was concerned with apparatus for establishing A and D which were stated as Corollaries 9.3 and 9.4. §4 identified criteria for the convergence of sequences of homeomorphisms and §5 introduced isotopy procedures to be used later.

§§6 and 7 were concerned with properties of s while §8 was concerned with properties of l_2 analogous to those of §7 for s. In this treatment, it appears as if l_2 is a simpler space to handle than is s, but in fact the procedure was designed for l_2 and then adapted to s since we were unable to adapt to l_2 a more natural simpler procedure for s.

Finally, in §9, the results of §§7 and 8 pushing individual points off s and l_2 were used to push $\bigcup C_i$ off s and $\bigcup E^i$ off l_2.

11. Some open questions related to s or l_2. Various topological properties of s or l_2 have been studied by Klee [13], [14], [15], Anderson [1], [2], [3], [4] and others. In various survey papers to appear in the Proceedings of the Symposium on Infinite Dimensional Topology (Annals of Mathematics Study), such properties are discussed. Here we list a number of open questions suggested conversationally or in print by one or more of several authors including Bessaga, Pelczynski, Klee, Fort, Borsuk, Eells, Palais, Henderson, Corson and others.

In this section we restrict ourselves to questions directly related to s or l_2 (and thus to separable metric spaces). See Bessaga [6] for many questions concerning homeomorphisms between various topological linear spaces.

Questions Concerning the Hilbert Cube as a Compactification of s. Let the Hilbert cube, I^∞, be written as $\prod_{i>0} I_i$ where for each $i>0$, I_i is the closed interval $[-1, 1]$. As in §3, we may regard $s=\prod_{i>0} I_i^o$ where for each $i>0$, I_i^o is the open interval $(-1, 1)$. Then $s\subset I^\infty$ and I^∞ is a compactification of s. Indeed, both s and $I^\infty\backslash s$ are dense in I^∞. A *β^*-homeomorphism* is a homeomorphism h of I^∞ onto itself such that $h(s)=s$ and a *β-homeomorphism* is a homeomorphism h of I^∞ onto itself such that $h(s)\supset s$. A closed set K in a space X is said to have *Property* Z if for every nonnull homotopically trivial open set U in X, $U\backslash K$ is nonnull and homotopically trivial.

(1) Let $K\subset I^\infty$. What are necessary and sufficient conditions on K (or on $I^\infty\backslash K$) in order that there exist a homeomorphism of I^∞ onto itself carrying K onto $B(I^\infty)$? Anderson has recently shown that if $K\supset B(I^\infty)$ then in order that such a homeomorphism exist it is necessary and sufficient that K be the countable union of closed sets with Property Z.

(2) A special case of (1) is the following. Let $s\subset M\subset I^\infty$. What are necessary and sufficient conditions on M in order for there to exist a β-homeomorphism h such that $h(s)=M$? In particular, is it necessary and sufficient that (a) $I^\infty\backslash M$ be the countable union of compact sets and (b) each compact subset of M have Property Z in M? (Conditions (a) and (b) are clearly necessary.)

(3) A different version of question (2) would ask simply for necessary and sufficient conditions that s and M be homeomorphic.

(4) Let s_f and I_f^∞ be the sets of all points of s and I^∞ with only finitely many nonzero coordinates. Characterize the subsets K of s (or I^∞) for which there exist homeomorphisms of s (or I^∞) onto itself carrying K onto s_f (or I_f^∞).

(5) Regarding $s\subset I^\infty$ it is easy to show that there exist homeomorphisms of s onto itself that cannot be extended to β^*-homeomorphisms, but the following question seems more interesting. For any homeomorphism h of s onto itself does there exist a homeomorphism f of s onto itself such that $f^{-1}hf$ can be extended to a β^*-homeomorphism?

Questions on Infinite Products. (6) Is every product of a compact absolute retract by I^∞ homeomorphic to I^∞?

(7) Is every countable infinite product of compact absolute retracts homeomorphic to I^∞?

A set is a topologically complete absolute retract if it admits a complete metric and is a retract of every metric space in which it is embedded as a closed set.

(8) Is every product of a topologically complete absolute retract by s homeomorphic to s?

(9) Is every countable infinite product of topologically complete

absolute retracts (with infinitely many noncompact factors) homeomorphic to s?

It is reasonable to weaken questions (6) to (8) by replacing "absolute retract" by "contractible finite complex" or a similar condition. In an as yet unpublished paper, Anderson has shown that any countable infinite product of dendrons is homeomorphic to I^∞. (A dendron is an acyclic locally connected one-dimensional continuum.) He feels it likely that the methods of the proof of this proposition can be modified to show that any countable infinite product of contractible finite complexes is homeomorphic to I^∞.

(10) Is every product of I^∞ by a finite connected complex homeomorphic to a product of I^∞ by some manifold-with-boundary?

(11) Is every product of s by a finite connected complex homeomorphic to a product of s by some manifold-with-boundary?

Questions on Topological Banach Manifolds. A number of authors have studied the differential topology of so-called Banach manifolds. For our purposes, we consider a *topological Banach manifold* (T.B.M.) to be a connected separable metric space locally like s—that is with an open covering of sets each homeomorphic to s. (In effect we are specifying that our Banach manifolds are modeled on some separable infinite dimensional Banach space.)

(12) Can every T.B.M. be imbedded in s as an open set in s?

(13) If a T.B.M. can be imbedded in s as an open subset of s, then the T.B.M. has a local compactification as a metric space admitting a cover by open sets homeomorphic to open subsets of I^∞. Furthermore the T.B.M. intersects such open sets in I^∞ in the manner induced by the open imbedding. As a weaker version of the imbedding problem of (12), we can ask if such a local compactification of each T.B.M. is possible.

(14) Is every contractible T.B.M. homeomorphic to s? What about the special case where the T.B.M. is homeomorphic to an open subset of s?

(15) More generally than in (14), are every two T.B.M.'s that are of the same homotopy type, homeomorphic to each other?

Questions on Unions of Two Sets. (16) Let $M=H\cup K$. If H, K, and $H\cap K$ are all homeomorphic to I^∞, is $M\sim I^\infty$?

(17) Let $M=H\cup K$. If H and K are both closed or both open and if H, K, and $H\cap K$ are all homeomorphic to s, is $M\sim s$?

(18) Let $M=H\cup K$. If H, K, and M are all homeomorphic to I^∞, is $H\cap K\sim I^\infty$?

(19) Let $M=H\cup K$. If H and K are both closed or both open and if H, K, and M are all homeomorphic to s, is $H\cap K\sim s$?

Questions on Spaces of Closed Subsets. (20) Is the space of all closed subsets (with the Hausdorff metric) of an n-cell (or of I^∞) homeomorphic to I^∞? This is not even known for $n=1$.

(21) Is the space of all subcontinua of an n-cell ($n>1$) or of I^∞ homeomorphic to I^∞?

Questions on Spaces of Homeomorphisms. (22) Is s homeomorphic to the space of all homeomorphisms of a disk onto itself which are the identity on the boundary of the disk? It is known that s is homeomorphic to the space of orientation preserving homeomorphisms of an interval onto itself.

(23) For any geometric n-sphere or n-cube X and any sufficiently small $\epsilon>0$, is s homeomorphic to the space of all homeomorphisms of X onto itself which are within ϵ of the identity? The answer is almost certainly *yes* for $n=1$.

(24) Is s homeomorphic to the space of all homeomorphisms of I^∞ onto itself?

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