THE VANISHING OF A THETA CONSTANT IS A PECULIAR PHENOMENON

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1. Introduction. The phenomenon in question is the following: (i) if a theta constant θ vanishes at a point t of Torelli space then its gradient (with respect to the coordinates on Torelli space) vanishes there, too; (ii) on the other hand, the locus $\theta=0$ through t is, generically, a hypersurface with tangent plane defined at t, in particular $\theta \not\equiv 0$ on Torelli space.

The reconciliation of (i) and (ii) results from (iii) near t one has $\theta=\Phi^k$, $k>1$ integral, and Φ analytic with nonvanishing gradient at t.

I would speculate that $k=2$, generically, i.e., the locus $\theta=0$ is really the locus $\sqrt{\theta}=0$.

In the next section I shall prove (i). (ii) is in the thesis of Dr. Farkas [**1**], while (iii) is an immediate consequence of (i) and (ii) and some standard algebra in several variables. The speculation on the value 2 for k stems from the appearance of those period relations that are known (Schottky). § 2 is a revision of the remarks in [**3**, pp. 35–37].

2. Definitions and proof of (i). Given a symmetric $g\times g$ complex matrix A with negative definite real part, one can form the Riemann theta function

$$\theta(u, A) = \sum_n \exp(n\cdot An + 2n\cdot u),$$

where n ranges over all integral column g-vectors, u is a column g-vector of complex numbers, and the dot signifies the usual inner product. If, in addition, one is given two column g-vectors ϵ and ϵ' whose entries are 0 or 1, one defines the first order theta function with binary characteristic

$$\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$$

by

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(u, A) = \theta(u+e, A)\exp\left(\frac{\epsilon}{2}\cdot\frac{A\epsilon}{2} + 2\frac{\epsilon\cdot u}{2} + 2\pi i\frac{\epsilon}{2}\cdot\frac{\epsilon'}{2}\right),$$

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where $e = \pi i\epsilon'/2 + A\epsilon/2$. Here e, and with it

$$\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$$

is defined to be even or odd according as $\epsilon \cdot \epsilon' \equiv 0, 1 \pmod 2$, and

$$\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(u, A)$$

is an even or odd function of u according as e is even or odd. In particular if e is even then

$$\partial\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(0, A)/\partial u_j = 0, \qquad j = 1, \cdots, g.$$

Now define the first order theta constant with characteristic

$$\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \text{ by } \theta_{\epsilon\epsilon'} = \theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(0, A).$$

Only the even case is of interest, since all odd theta constants are, of course, identically 0 in A.

Let S be a compact Riemann surface of genus g and let (γ, δ) be a canonical homology basis on S. Further, let $d\xi_i$, $i = 1, \cdots, g$ be a set of abelian differentials of first kind with Riemann's normalization with respect to (γ, δ):

$$\int_{\gamma k} d\xi_h = \pi i \delta_{hk}.$$

Observe that $d\xi_j = \pi i d\zeta_j$, $j = 1, \cdots, g$, when the $d\zeta$ are the conventional normal differentials of first kind as defined, e.g., in [3].

Now define, given $P \in S$ fixed and $Q \in S$,

$$\theta_{\epsilon\epsilon'}(Q) = \theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(u, A),$$

$$A = (a_{ij}), \qquad a_{ij} = \int_{\delta_j} d\xi_i,$$

$$u = \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_g \end{pmatrix}, \qquad u_i = \int_P^Q d\xi_i, \quad i, j = 1, \cdots, g,$$

where the path from P to Q is arbitrary but fixed and the same for each i. This is multivalued, but a zero is well defined.

If e is even, then the Riemann vanishing theorem (in view of the vanishing of the first partials of

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(u, A)$$

at $u=0$) ([2, Theorem 6 or 8]) implies that

$$\epsilon_{\epsilon\epsilon'} = 0 \Rightarrow \epsilon_{\epsilon\epsilon'}(Q) \equiv 0. \tag{1}$$

In particular if Q lies in some parameter disk about P and has parameter value z (P has the value $z=0$) one finds, setting $P=Q$ ($z=0$),

$$0 = \frac{d^2\theta\epsilon\epsilon'(P)}{dz^2} = \sum_{i,j} \frac{\partial^2\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0, A)}{\partial u_i \partial u_j} \frac{d\xi_i(P)}{dz} \frac{d\xi_j(P)}{dz}, \tag{2}$$

where I recall that $P\in S$ is arbitrary.

On the other hand, S and (γ, δ) specify the point $t=\{S, S, 1\}\in\mathfrak{T}^g(S)$ where the transition to any other Torelli space and/or equivalent point on it is easily made by the rules in [3]. I should now like to compute the gradient of $\theta_{\epsilon\epsilon'}$ at t. Using Prescription II of [3] with the obvious changes wrought by the substitution of a_{ij} and $d\xi_i$ for π_{ij} and $d\zeta_i$ one finds

$$\begin{aligned}
\left.\frac{\partial\theta_{\epsilon\epsilon'}}{\partial c_\alpha}\right|_{c=0} &= -\frac{1}{\pi}\sum_{i\le j}\left.\frac{\partial\theta_{\epsilon\epsilon'}}{\partial a_{ij}}\frac{\partial a_{ij}}{\partial e_\alpha}\right|_{c=0} \\
&= -\frac{1}{\pi}\sum_{i\le j}\frac{\partial\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0, A)}{\partial a_{ij}}\left(\frac{d\xi_i}{dz}\frac{d\xi_j}{dz}, \mu_\alpha\right) \\
&= -\frac{1}{\pi}\left(\sum_{i\le j}\frac{\partial\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0, A)}{\partial a_{ij}}\frac{d\xi_j}{dz}\frac{d\xi_j}{dz}, \mu_\alpha\right).
\end{aligned} \tag{3}$$

But I claim that the sum on the right side of (2) is 4 times the sum in parenthesis in the last expression in (3). If one grants this then (1) and (2) imply immediately that

$$\theta_{\epsilon\epsilon'} = 0 \Rightarrow \left.\frac{\partial\theta_{\epsilon\epsilon'}}{\partial c_\alpha}\right|_{c=0} = 0, \qquad \alpha = 1, \cdots, 3g-3, \tag{4}$$

which is statement (i) of §1.

To establish the remaining link I invoke the "heat equations"

$$2\frac{\partial\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}}{\partial a_{ij}}(u, A) = \frac{\partial^2\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}}{\partial u_i \partial u_j}(u, A), \qquad i \neq j,$$

$$= \frac{1}{2}\,\frac{\partial^2\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}}{\partial u_i^2}(u, A), \qquad i = j,$$

in which I put $u=0$, and then I split the sum $\sum_{i,j}$ in (2) into $2\sum_{i<j}+\sum_{i=j}$.

Bibliography

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