BOUNDS FOR LINEAR FUNCTIONALS[1]

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We shall obtain upper and lower bounds for certain functionals associated with linear equations involving positive operators. Attention is focused on these functionals because of their considerable physical significance in applications. A bound from one side is furnished by the usual variational principle. For boundary value problems the reciprocal variational principle introduced by Friedrichs, and later modified by Diaz, provides a complementary bound. In the present article we extend these ideas to an integral equation over a domain E. Our procedure requires information (which is often available) for the same integral equation over some larger domain E'. This approach bears resemblance to the one used by Weinstein and Aronszajn in a series of papers dealing with eigenvalue problems.

Suppose then that we wish to estimate

$$I = \int_E f(x)u(x)dx$$

where

$$u(x) + \int_E k(x, y)u(y)dy = f(x), \qquad x \in E. \tag{1}$$

We assume that we know how to solve the integral equation

$$Az = z(x) + \int_{E'} k(x, y)z(y)dy = h(x), \qquad x \in E', \tag{2}$$

for some domain $E' \supset E$.

The situation described above occurs frequently in applications. For instance, if the domains are one-dimensional and the kernel is a difference kernel $k(x-y)$, then the integral equation (2) is easily solved if (a) $k(x)$ has period T and E' is an interval of length T, or (b) $k(x)$ is Fourier transformable and E' is the whole real axis.

Since the method we employ is not restricted to integral equations, we describe it in a slightly more abstract setting.

Let A be a real, self-adjoint, positive operator on the space of real L_2 functions over E'. The usual inner product of two functions $v(x)$

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and $w(x)$ is written $\langle v, w\rangle$. We denote by P the projection operator defined by

$$Ph = h(x), \qquad x \in E,$$
$$= 0, \qquad x \in (E' - E).$$

The integral equation (1) can then be rewritten

$$PAu = f, \quad \text{with} \quad Pu = u, \qquad Pf = f. \tag{3}$$

If u has been found, then Au can be calculated for all $x \in E'$ and we have

$$Au = f + g \quad \text{with} \quad Pg = 0. \tag{4}$$

The equation (2) takes the form $Az = h$, where A^{-1} is regarded as known.

We wish to estimate $I = \langle f, u\rangle = \langle PAu, u\rangle = \langle Au, u\rangle$. It is convenient to introduce a new inner product $[v, w] = \langle v, Aw\rangle$, in terms of which $I = [u, u]$.

In what follows we need the Schwarz inequality

$$[u, u] \geqq [v, u]^2/[v, v] \quad \text{for all } v \ni [v, v] \neq 0,$$

and Bessel's inequality in the simple form

$$[u, u] \leqq [v, v], \quad \text{for all } v \ni [v - u, u] = 0.$$

From the Schwarz inequality we find the well-known lower bound

$$I \geqq \langle f, v\rangle^2/\langle v, Av\rangle \quad \text{for all } v \ni v = Pv. \tag{5}$$

It is clear that the maximum is actually attained for $v = cu$, where c is any nonzero constant.

To apply the Bessel inequality, we must first characterize functions v for which $[v - u, u] = \langle A(v - u), Pu\rangle = 0$. This condition is certainly satisfied if $PA(u - v) = 0$ or $PAv = f$. We should observe that this last equation is not identical with (3) since we do *not* require $Pv = v$. The Bessel inequality then yields the following upper bound:

$$I \leqq \langle v, Av\rangle \quad \text{for all } v \ni PAv = f. \tag{6}$$

By choosing $v = u$, the minimum in (6) is obviously attained.

We now rewrite (6) with a view toward the application of the Rayleigh-Ritz method. Let v_0 be an arbitrary fixed function such that $PAv_0 = f$, that is,

$$Av_0 = f + g_0, \quad \text{with } Pg_0 = 0.$$

Since A^{-1} is known, this equation can be solved for any g_0, but convenience or physical considerations will usually dictate the choice of g_0. We then define

$$I_0 = \langle v_0, Av_0\rangle = \langle v_0, f + g_0\rangle.$$

Any function v for which $PAv=f$ can be written $v=v_0+w$, with $Aw=q$ and $Pq=0$. Substituting in (6), we find

$$I \leqq I_0 + 2\langle v_0, q\rangle + \langle q, A^{-1}q\rangle;\ Pq = 0. \tag{7}$$

We note that the right side of this inequality reduces to I when q is chosen equal to $g-g_0$, where g is defined from (4).

To apply the Rayleigh-Ritz procedure to (7), we introduce an independent set of functions $\psi_1, \cdots, \psi_n$ with the property $P\psi_k=0$, $k=1, \cdots, n$. Then, for any choice of $c_1, \cdots, c_n$,

$$I \leqq I_0 + 2\left\langle v_0, \sum_{k=1}^{n} c_k\psi_k\right\rangle + \left\langle \sum_{k=1}^{n} c_k\psi_k, A^{-1}\left(\sum_{j=1}^{n} c_j\psi_j\right)\right\rangle.$$

The values of the coefficients which minimize the right side of the above inequality are obtained from the Galerkin equations

$$\langle v_0, \psi_k\rangle = -\sum_{j=1}^{n} c_j\langle A^{-1}\psi_j, \psi_k\rangle; \qquad k = 1, \cdots, n. \tag{8}$$

The corresponding approximation, call it q^*, to $g-g_0$ is then

$$q^* = \sum_{k=1}^{n} c_k\psi_k, \tag{9}$$

where the $\{c_k\}$ are calculated from (8).

We observe that q^* satisfies the reciprocity principle

$$\langle v_0, q^*\rangle = -\langle A^{-1}q^*, q^*\rangle$$

so that

$$I \leqq I_0 - \langle q^*, A^{-1}q^*\rangle = I_0 + \langle v_0, q^*\rangle.$$

If we use a one term approximation $q^*=c\psi$, we find

$$I \leqq I_0 - \langle v_0, \psi\rangle^2/\langle \psi, A^{-1}\psi\rangle; \qquad P\psi = 0.$$

In conjunction with (5), we have

$$\langle \phi, f\rangle^2/\langle \phi, A\phi\rangle \leqq I \leqq I_0 - \langle v_0, \psi\rangle^2/\langle \psi, A^{-1}\psi\rangle,$$

where $P\phi=\phi$ and $P\psi=0$. In practice, the trial functions ϕ and ψ

should be chosen to be reasonable approximations to u and $g-g_0$, respectively.

Applications to physical problems will be described elsewhere.

Bibliography

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