SURGERY ON PIECEWISE LINEAR MANIFOLDS AND APPLICATIONS

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1. Introduction and statement of results. In this note we indicate a method of performing surgery on piecewise linear (=PL) manifolds, and show how to prove piecewise linear analogs of theorems on the homotopy type and classification of smooth manifolds[2] (Browder [1], Novikov [10], Wall [13]).

The basic principles are two: to use normal microbundles instead of normal vector bundles, and to put a differential structure σ on a neighborhood V of an embedded sphere $S \subset M$ that represents a homotopy class we wish to kill. Then smooth ambient surgery can be performed on V_σ, and the resulting cobordism triangulated.

Let M_1, M_2 be closed PL n-manifolds embedded in S^{n+k} with normal microbundles ν_1, ν_2. A *normal equivalence* $b\colon (M_1, \nu_1) \to (M_2, \nu_2)$ is a microbundle equivalence $b\colon \nu_1 \to \nu_2$ covering a homotopy equivalence $M_1 \to M_2$.

Let $T(\nu_i)$ be the Thom complex of ν_i (see [12]), and let $c_i \in \pi_{n+k}T(\nu_i)$ be the homotopy class of the collapsing map $S^{n+k} \to T(\nu_i)$. We call c_i a *normal invariant for* M_i. If $\partial M \neq 0$, a similar construction defines a normal invariant for M as an element in $\pi_{n+k}(T(\nu_M), T(\nu_M | \partial M))$.

THEOREM 1. *Let X be a 1-connected polyhedron satisfying Poincaré duality in a dimension $n \geqq 5$. Let ξ be a* PL *k-microbundle over X, and let $\alpha \in \pi_{n+k}T(\xi)$ be such that $h(\alpha) = \Phi(g)$, where $h\colon \pi_{n+k}T(\xi) \to H_{n+k}T(\xi)$ is the Hurewicz homomorphism, $\Phi\colon H_n(X) \to H_{n+k}T(\xi)$ is the Thom isomorphism, and $g \in H_n(X)$ is a generator. Assume $k \geqq n$. Then X has the homotopy type of a closed* PL *n-manifold $M \subset S^{n+k}$ such that*

(a) *If n is odd, or if $n = 4q$ and the signature of X is $\langle L_q(p_1(\xi), \cdots, p_q(\xi), g)\rangle$, then M has a normal microbundle induced from ξ, and α is a normal invariant of M;*

(b) *If n is even, M-$\{point\}$ has a normal microbundle induced from ξ.*

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[2] We are informed that some of our results have been obtained independently by R. Lashof and M. Rothenberg.

Theorem 1 is the PL analog of [**1**]; see also [**10**].

THEOREM 2. *Let* M_1, M_2 *be* PL *closed* 1-*connected* n-*manifolds* $n \geqq 5$. *Then* M_1 *and* M_2 *are combinatorially equivalent if and only if there are normal microbundles* ν_i $(i=1, 2)$ *of embeddings* $M_i \subset S^{n+k}$, *with normal invariants* $c_i \in \pi_{n+k}T(\nu_i)$, *and a normal equivalence* $b\colon (M_1, \nu_1) \to (M_2, \nu_2)$ *such that* $T(b)_*(c_1) = c_2$.

Theorem 2 is the PL analog of a theorem of Novikov [**10**].

COROLLARY. *Let* M *be a* PL *closed* 1-*connected* n-*manifold*, $n \geqq 5$. *Suppose the natural map* $k_{\mathrm{PL}}(M) \to k_{\mathrm{Top}}(M)$ *is injective and that* $k_{\mathrm{PL}}(\Sigma M) \to k_{\mathrm{Top}}(\Sigma M)$ *is surjective* (see [**8**] and [**9**]) *where* ΣM *is the suspension of* M. *Then the* PL *structure on the underlying topological manifold* M *is unique up to isomorphism.*

PROOF. Let ν_1, ν_2 be normal microbundles of two PL structures M_1, M_2 on M. By the stable uniqueness of a topological normal microbundle of M [**8**], and the injectivity of $k_{\mathrm{PL}}(M) \to k_{\mathrm{Top}}(M)$, it follows that ν_1 and ν_2 are stably equivalent as PL microbundles. Let $c_i \in \pi_{n+k}T(\nu_i)$ be the normal invariant of M_i. Since M_1 and M_2 are the same topological manifold, it follows that (for sufficiently large k) there is a topological microbundle equivalence $b\colon \nu_1 \to \nu_2$ such that $T(b)_*(c_1) = c_2$. (The stable tubular neighborhood theorem [**4**], [**7**] is needed.) Using the surjectivity of $k_{\mathrm{PL}}(\Sigma M) \to k_{\mathrm{Top}}(\Sigma M)$ we can choose b to be a PL microbundle equivalence. The Corollary follows from Theorem 2.

THEOREM 3. *Let* (X, A) *be a polyhedral pair with both* X *and* A 1-*connected, satisfying Poincaré duality in a dimension* $n \geqq 6$. *Let* ξ *be a* PL k-*microbundle over* X *with* $k > n$, *let* $e \in H_n(X, A)$ *be a generator, and suppose there exists* $\beta \in \pi_{n+k}(T(\xi), T(\xi|A))$ *such that* $h(\beta) = \Phi(e)$. *Then* (X, A) *is homotopy equivalent to* PL *manifold with boundary* $(M, \partial M)$ *having a normal microbundle induced from* ξ, *and having* β *for a normal invariant. Moreover,* M *is unique up to* PL *homeomorphism.*

This is the PL analog of a result of Wall [**13**].

2. **Proofs of theorems.** We indicate the modification in the proofs of the analogous smooth theorems that are required in the PL case.

To prove Theorem 1, by using the transverse regularity theorem of Williamson [12] we may assume that there is a PL closed n-manifold $N \subset S^{n+k}$ such that:

(i) if $\bar{f}: S^{n+k} \to T(\xi)$ represents α, then $\bar{f}^{-1}(X) = N$;

(ii) if $\bar{f} | N = f$, then $f^*\xi = \nu$, the normal microbundle of N in S^{n+k};

(iii) $f: N \to X$ has degree 1.

(See [1].)

MAIN LEMMA. *Let $S \subset N$ be a* PL *embedded p-sphere, $p < n/2$, such that $f | S: S \to X$ is null homotopic. Then there exists a* PL *surgery killing the homotopy class of S. If N' is the resulting n-manifold the trace of the surgery (an elementary* PL *cobordism K between N and N') can be embedded in $S^{n+k} \times I$ with $K \cap (S^{n+k} \times 0) = N = N \times 0$ and $K \cap (S^{n+k} \times 1) = N'$. Moreover, K has a* PL *normal microbundle η in $S^{n+k} \times I$ with $\eta = g^*\xi$, where $g: K \to X$ extends $f: N \to X$.*

PROOF. Let $U \subset N$ be an open regular neighborhood of S. Then $f^*\xi | U = \nu | U$ is trivial because $f | U$ is null homotopic. Therefore there is a PL embedding $\phi: U \times R^k \to S^{n+k}$ such that $\phi(x, 0) = x$ and $\phi^{-1}N = U \times 0$. By the product theorem of [5], the smoothing of $U \times R^k$ induced by ϕ is concordant to a product smoothing. In fact, there is an open neighborhood V of S in N with $\bar{V} \subset U$, a smoothing σ of V, and a piecewise differentiable isotopy $\phi_t: U \times R^k \to S^{n+k}$ such that

(i) $\phi_0 = \phi$,

(ii) $\phi_t = \phi$ outside $V \times R^k$,

(iii) $\phi_1 | V \times D^k$ is a smooth embedding $V_\sigma \times D^k \to S^{n+k}$.

Observe now that $\phi_1(V_\sigma \times 0)$ is a smooth submanifold of S^{n+k} and ϕ_1 provides a trivialization of its normal vector bundle. Let $V' \subset V_\sigma$ be a smooth closed neighborhood of S, and put $W_0 = \phi_1(V' \times 0)$. Let $W_1 \subset S^{n+k}$ be the smooth submanifold obtained from W_0 by a smooth surgery killing the homotopy class of $\phi(S \times 0)$. By Haefliger [2] the trace of the surgery is a cobordism L between W_0 and W_1 smoothly embedded in $S^{n+k} \times I$ such that $\partial L = W_0 \times 0 \cup (\partial W_0) \times I \cup W_1 \times 1$, and such that the embedding is the product embedding in a neighborhood of $\partial W_0 \times I$. Furthermore, the map $f': W_0 \times 0 \cup (\partial W_0) \times I \to X$, defined to be the composition

$$(W_0 \times 0) \cup (\partial W_0) \times I \to W_0 \xrightarrow{\phi_1^{-1}} N \to X$$

extends to $f'': L \to X$ such that $f''^*\xi$ is the normal bundle of L in $S^{n+k} \times I$.

The cobordism L and the product cobordism $(N-\text{int } V')\times I$ fit together to form a cobordism $K_1\subset S^{n+k}\times I$ between $N\times 0$ and $((N-\text{int } V')\cup W_1)\times 1$. The composition

$$(N - \text{int } V') \times I \to N \xrightarrow{f} X$$

and $f'': L\to X$ fit together to give a map $g: K_1\to X$. The microbundle ν extends to a microbundle η over K_1 that coincides with ν over $N\times 0$, with $\nu\times I$ over $(N-\text{int } V')\times I$, and such that ϕ_1 is a trivialization of $\eta | W_1\times 1$. In fact, $\eta=g^*\xi$. The isotopy ϕ_t provides an embedding $G: E\eta\to S^{n+k}\times I$ of the total space η which is the identity on $E\nu$. Consider G as a smooth triangulation of an open subset of $S^{n+k}\times I$. Whitehead's triangulation theorems show that there is a neighborhood E_0 of the zero section of η and a homeomorphism H of $S^{n+k}\times I$ such that $HG| E_0$ is PL, and $H| S^{n+k}\times 0$ is the identity. Thus $K=HG(K_1)$ is the desired cobordism. This completes the proof of the Main Lemma.

The proof of Theorem 1 proceeds as in the smooth case if n is odd. If n is even, we proceed until we have an N such that $f: N\to X$ is an isomorphism in homotopy below the middle dimension. Following the procedure of the proof of the main lemma, we find just as in the smooth case that the obstruction c to surgery is a signature or Kervaire-Arf invariant of the intersection quadratic form on the kernel K_r of f_* in $H_r(N)$, $2r=n$. If the signature of X is as in (a) of Theorem 1, then $c=0$; otherwise $c\equiv 0$ mod 8. (To see this, recall that a nonsingular quadratic form taking only even values has signature divisible by 8. It suffices to prove $x\# x=0$ for $x\in\ker(f_*| H_r(N; Z_2))$. If $P:H^*(N;Z_2)\to H_*(N;Z_2)$ is Poincaré duality, then $x\# y=\langle P^{-1}x\cup P^{-1}y, N\rangle$ for $x, y\in H_r(N; Z_2)$. Let $P^{-1}x=z$. Then $x\# x=\langle \text{Sq } z, N\rangle=\langle z\cup U_N, N\rangle$ where $U_N\in H^*(N)$ is the total Wu class. Since $\text{Sq}^{-1} U_N=W_N$ (the total Stiefel-Whitney class of N), if we define $U_X=\text{Sq}^{-1}W(\xi)^{-1}$ it follows that $U_N=f^*U_X$, and $x\#x=\langle Z\cup f^*U_X, N\rangle=x\#Pf^*U_X$. By [1], K_r is orthogonal to $Pf^*(H^*(X))$. Hence $x\#x=0$.)

There exists an oriented PL closed $(r-1)$-connected $2r$-manifold P with signature -8 if $r=2q$, and with Kervaire-Arf invariant 1 if $r=2q+1$. Moreover P-$\{\text{point}\}$ is parallelizably smoothable. It follows [3] that there is a PL embedding $P\subset S^{2r+2}$ having a trivial normal bundle on P_0 (the complement of a highest dimensional cell). Therefore the connected sum $N\# P$ embeds in S^{n+k} with a normal microbundle ν' on $(N\# P)_0$ which coincides with the normal microbundle ν of N on N_0, and which is trivial on the rest of $(N\# P)_0$.

Let $N' = N \# P$ if $r = 2q+1$, and let N' be the connected sum of N with $c/8$ copies of P if $r = 2q$. Define $f': N' \to X$ by $f' | N_0 = f$, and $f | N' - N_0$ constant. Since $\nu' | N' - N_0$ is trivial, f' is covered by a microbundle map $\nu' \to \xi$. The obstruction to surgery on N' now vanishes. Hence by surgery we obtain a manifold $M \subset S^{n+k}$ with a normal microbundle ν on M_0 and a homotopy equivalence $f: M \to X$ such that $f | M_0$ is covered by a microbundle map $\nu \to \xi$.

Alternatively, in the middle dimension we could use the method of [**14**].

Theorem 2 is proved in a similar way, using the same trick to extend Novikov's proof to the PL case. Since for $n \geqq 5$ any PL homotopy sphere T is a combinatorial sphere (Smale [**11**]), the conclusion of the smooth case, that $M_1 \# T = M_2$ becomes $M_1 = M_2$ in the PL case.

For Theorem 3 we imitate the proof of Theorem 2 of Wall [**13**] with the following modification of the immersion argument of [**13**]. Given a PL map $f: D^{k+1} \to M^{2k+1}$ (in the notation of [**13**]), assume that f has generic singularities. It follows that $H^i f(D^{k+1}) = 0$ for $i > 2$. Since $\Gamma_i = 0$ for $i \leqq 2$, it follows from [**5**] that a neighborhood V of $f(D^{k+1})$ in M^{2k+1} has a smoothing σ. Then we approximate f by a smooth map into V_σ and proceed as in [**13**].

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