ON REGULAR NEIGHBORHOODS OF SPHERES

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Consider the following two conjectures:

$C(n)$: (The combinatorial Schoenflies conjecture.) A combinatorial $(n-1)$-sphere on a combinatorial n-sphere decomposes the latter into two combinatorial n-cells.

$D(n)$: Let W^n be an orientable combinatorial manifold without boundary and let M^{n-1} be a closed orientable combinatorial manifold imbedded piecewise linearly in W^n. Let U be a regular neighborhood of M^{n-1} in W^n. Then there exists a piecewise linear homeomorphism $h: M^{n-1}\times[-1;1]\to U$ such that

(1) $$h(x, 0) = x,$$

(2) $$h \text{ is onto.}$$

It is easily seen that $D(n)$ implies $C(n)$ for all $n\neq 4$ by using the Hauptvermutung for combinatorial cells and spheres [**10**]. In [**8**], Noguchi shows that $C(1), C(2), \cdots, C(n)$ imply $D(n+1)$. By using the fact that a compact component of the boundary of a combinatorial manifold is combinatorially collared [**9**], [**11**], it is easily shown that $C(n)$ implies $D(n+1)$. However it is possible to prove a weaker version of $D(n+1)$ without the use of $C(n)$ for the special case when W, M are spheres.

THEOREM. *Let $\sum^n$ $(n\neq 4)$ be a combinatorial sphere embedded piecewise linearly in the combinatorial sphere S^{n+1}. Let U be a regular neighborhood of $\sum^n$ in S^{n+1}. Then there exists a piecewise linear homeomorphism $h: \sum^n\times[-1;1]\to S^{n+1}$ such that $h(\sum^n\times[-1;1])=U$.*

PROOF. (For definitions of terms used see [**11**].) Since $C(i)$, $i=1, 2, 3$, is valid [**1**], [**6**], it follows from the remarks above that the theorem is true for $n<4$. Suppose $n>4$.

Since $\sum^n$ is a deformation retract of U, the ith integral homology groups of $\sum^n$ and U are isomorphic for all i. It follows then from Alexander duality and the unicoherence of the sphere that the closure of $S^{n+1}-U$, $\mathrm{Cl}(S^{n+1}-U)$, is the union of two connected closed sets,

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D_1, D_2 with a connected boundary T_1, T_2 respectively. Since U is a combinatorial manifold, from [2], we have that each D_i is a combinatorial manifold. Similarly, $S^{n+1}-\sum^n=R_1\cup R_2$ where $D_i\subset R_i$ and $\mathrm{Cl}\, R_1\cap \mathrm{Cl}\, R_2=\sum^n$. By either [3] or [7], $\mathrm{Cl}\, R_1$ and $\mathrm{Cl}\, R_2$ are topological $(n+1)$-cells.

We want to show that each T_i is simply connected. Let $f\colon S^1\to T_1$ be a continuous map of the 1-sphere into T_1. By the simplicial approximation theorem, we may assume f is piecewise linear. Since U is simply connected (for it is of the same homotopy type as $\sum^n$), $f(S^1)$ bounds a disk N in U. We may assume N is polyhedral and in general position with respect to $\sum^n$. Then if $N\cap\sum^n\neq\emptyset$, $N\cap\sum^n$ is a finite collection of simple closed curves. Since $\sum^n$ is simply connected, we can suppose that N lies in $U\cap \mathrm{Cl}\, R_1$; for by the usual alteration techniques, see, for example, [4], we can replace N by a disk which is bounded by $f(S^1)$ and lies in $U\cap\mathrm{Cl}\, R_1$. By using the collar of the boundary of $\mathrm{Cl}\, R_1$, we can assume that $N\cap\ \sum^n=\emptyset$. Since $U-\sum^n=(T_1\cup T_2)\times[0,1)$, we can then push N into T_1.

Since $D_i\cup U\searrow \mathrm{Cl}\, R_i$, $i=1, 2$, it follows that each $D_i\cup U$ is contractible and hence from the fact that each T_i is bicollared and from duality, each D_i has homology groups of a point. Since each T_i is simply connected it follows from a similar argument as above that each D_i is simply-connected. From the Hurewicz Isomorphism Theorem, it follows that each D_i is contractible. Hence from [10], we have that each D_i is a combinatorial $(n+1)$-cell. From [2], each $\mathrm{Cl}(S^{n+1}-D_i)$ is a combinatorial $(n+1)$-cell. Then $U=\mathrm{Cl}(\mathrm{Cl}(S^{n+1}-D_1)-D_2)$ is piecewise linear homeomorphic to $\sum^n\times[-1;1]$ [11].

REMARKS. Attempts to prove the above theorem for manifolds not spheres by the techniques of Noguchi fail because of the missing dimension $n=4$. From [5], it follows that $T_1\times(0;1)$ is topologically homeomorphic to $S^n\times(0;1)$, but otherwise it is unknown to the author whether T_1 is a topological 4-sphere in the case $n=4$.

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