THE DISTRIBUTION OF THE SUM OF DIGITS (mod p)

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Let s_n be the sum of the digits of n written in the base $b>1$. S. Ulam has asked (for $b=10$) whether the number of $n<x$ for which $s_n \equiv n \equiv 0 \pmod{13}$ is asymptotically $x/13^2$. His question is answered here affirmatively by the following theorem.

THEOREM. *Let p be a prime such that $p \nmid (b-1)$, and let a and c be any residues* mod p. *If $N(x)$ is the number of $n<x$ for which $n\equiv a \pmod{p}$ and $s_n \equiv c \pmod{p}$, then*

$$\lim_{x\to\infty} \frac{N(x)}{x} = \frac{1}{p^2}\cdot$$

PROOF. Let $x=d_0+d_1b+d_2b^2+\cdots+d_kb^k$, with $0\leqq d_m<b$, $d_k>0$. For $j\geqq 0$, define

$$f_j(u,v) = 1 + uv^{b^j} + u^2v^{2b^j} + \cdots + u^{b-1}v^{(b-1)b^j}.$$

Also, let $A(i,n)=1$ if $0\leqq n<x$ and $s_n=i$, $A(i,n)=0$ otherwise. If

$$f(u,v) = \sum_{i,n} A(i,n)u^iv^n,$$

then, writing $\omega=\exp(2\pi i/p)$, we have

$$N(x) = \frac{1}{p^2}\sum_{g,h=0}^{p-1} \omega^{-cg-ah} f(\omega^g,\omega^h). \tag{1}$$

If $0\leqq n<x$, we may write, uniquely,

$$n = d_0' + d_1'b + \cdots + d_{m-1}'b^{m-1} + tb^m + d_{m+1}b^{m+1} + \cdots + d_kb^k, \tag{2}$$

with $0\leqq d_j'<b$ $(j=0,1,\cdots,m-1)$, $0\leqq t<d_m$, and $m=0,1,\cdots,k$. Splitting the generating function according to (2), we have

$$f(u,v) = \sum_{m=0}^{k}\left\{\prod_{r=m+1}^{k} u^{d_r}v^{d_rb^r}\right\}\sum_{t=0}^{d_m-1} u^tv^{tb^m}\prod_{j=0}^{m-1} f_j(u,v), \tag{3}$$

where an empty sum is 0, an empty product 1. Observe that $f(1,1) = x$, so

$$N(x) = \frac{x}{p^2} + \frac{1}{p^2}\sum_{(g,h)\neq(0,0)} \omega^{-cg-ah} f(\omega^g,\omega^h). \tag{4}$$

It will be sufficient, therefore, to show that $f(\omega^g, \omega^h) = o(x)$ if $(g, h) \neq (0, 0)$.

Now observe that

$$(5) \qquad |f_j(\omega^g, \omega^h)| \leqq b,$$

and that equality holds if and only if

$$(6) \qquad g + hb^j \equiv 0 \pmod{p}.$$

Also, if (6) does not hold, then

$$(7) \qquad |f_j(\omega^g, \omega^h)| \leqq \lambda b,$$

where $\lambda < 1$ depends only on p and b. In fact,

$$(8) \qquad \lambda = \frac{|\sin \pi b/p|}{b \sin \pi/p}.$$

To estimate the error in (4), we distinguish two cases. First, suppose that $p|b$. Then $f_0(\omega^g, \omega^h) = 0$ unless $g + h \equiv 0 \pmod{p}$, and $f_1(\omega^g, \omega^h) = 0$ unless $g \equiv 0 \pmod{p}$. Since every term with $m > 1$ in (3) contains the factor $f_0 f_1$, we have

$$|f(\omega^g, \omega^h)| \leqq d_0 + d_1 b < b^2$$

when $(g, h) \neq (0, 0)$. In this case, therefore,

$$(9) \qquad \left| N(x) - \frac{x}{p^2} \right| < b^2.$$

Next, suppose that $p \nmid b$. For a given (g, h), if (6) holds for j and $j+1$, then

$$hb^j(b - 1) \equiv 0 \pmod{p},$$

so $h \equiv 0 \pmod{p}$, therefore $g \equiv 0 \pmod{p}$. Hence, if $(g, h) \neq (0, 0)$, the mth summand in (3) contains at least $[m/2]$ factors f_j for which (6) fails and (7) holds. Thus

$$(10) \qquad |f(\omega^g, \omega^h)| \leqq \sum_{m=0}^{k} d_m b^m \lambda^{[m/2]} \leqq b\lambda^{-1/2} \sum_{m=0}^{k} (b\lambda^{1/2})^m = O(x\lambda^{k/2}).$$

This completes the proof. [*Note*: The estimate in (10) can be improved to yield the exponent $k(1 - 1/\mu)$, where μ is the exponent to which b belongs mod p.]

We remark that for distinct primes p, q, the residues of $n \pmod{p}$ and $s_n \pmod{q}$ are asymptotically independent. The proof is simpler than the one given above, and there are no exceptional cases.

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