AVERAGES OF CONTINUOUS FUNCTIONS ON COUNTABLE SPACES[1]

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Introduction. Let $X=\{x_1, x_2, \cdots\}$ be a countably infinite topological space; then the space $C^*(X)$ of all bounded real-valued continuous functions f may be regarded as a space of sequences $(f(x_1), f(x_2), \cdots)$. It is well known [7, p. 54] that no regular (Toeplitz) matrix can sum all bounded sequences. On the other hand, if $(x_1, x_2, \cdots)$ converges in X (to x_m), then every regular matrix sums all f in $C^*(X)$ (to $f(x_m)$).

The main result of this paper is that if a regular matrix sums all f in $C^*(X)$ then it sums f to $\sum \alpha_n f(x_n)$, for some absolutely convergent series $\sum \alpha_n$. We use this to show that no regular matrix can sum all of $C^*(X)$ if X is extremally disconnected (the closure of every open set is open). This extends a theorem of W. Rudin [6], which has an equivalent hypothesis (X is embeddable in the Stone-Čech compactification βN of a discrete space) and concludes that not all f in $C^*(X)$ are Cesàro summable.

For any continuous linear functional ϕ on $C^*(X)$ one has a ("Riesz") representation $\phi(f)=\int f d\mu$, where μ is a Radon measure on βX. Our main result is just that X supports μ; μ is forced to be atomic since X is countable. We show further that X has a subset T, the set of *heavy points*, such that the functionals we are concerned with correspond exactly to measures μ supported by T with $\mu(T)=1$. Our knowledge of T is limited; it will be summarized elsewhere.

1. Representation. It is well known [7, p. 57] that a matrix $A=(a_{ij})$ defines a regular summability method if and only if it satisfies the conditions (1) $\sum_j a_{ij}=1+o(1)$, (2) $\sum_j |a_{ij}|$ is uniformly bounded, and (3) for each j, $a_{ij}\to 0$.

For all the present results on real-valued functions, we may assume without loss of generality that *our topological spaces are completely regular*. Then each countable space X has a base of open-and-closed sets, and each $f\in C^*(X)$ is a uniform limit of linear combinations of characteristic functions of these basic sets.

Suppose that A is a regular matrix such that $\phi_A(f) =\lim_{i\to\infty} \sum_j a_{ij}f(x_j)$ exists for each $f\in C^*(X)$. For each open-and-closed subset U of X, let c_U denote its characteristic function, and let

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$a(U)=\phi_A(c_U)$. Also, let $b(U)=\sup \sum |a(V_i)|$, where $\{V_i\}$ ranges over all finite disjoint families of open-and-closed subsets of U. A bound for $\sum_j |a_{ij}|$ is a bound for $b(U)$; also a and b are finitely additive.

For each point $x\in X$, let $\alpha(x)$ be the limit of $a(U)$ over the filter base $\mathfrak{U}_x$ of open-and-closed neighborhoods of x. It exists since the monotone function $b(U)$ converges, which implies that

$$\lim\{b(U \sim U')\colon U,\, U' \in \mathfrak{U}_x,\, U' \subset U\} = 0,$$

so $|a(U)-a(U')| = |a(U\sim U')| \leqq b(U\sim U')$.

LEMMA. *The series* $\sum \alpha(x_n)$ *is absolutely convergent with sum* 1.

PROOF. For any $\epsilon>0$, there exist $U_n\in\mathfrak{U}_{x_n}$, for $n=1, 2, \cdots$, such that for any $V_n\in\mathfrak{U}_{x_n}$ satisfying $V_n\subset U_n$, $\sum b(U_n\sim V_n)<\epsilon$. Thus $|\sum a(U_n)-\sum\alpha(x_n)|\leqq\epsilon$, and with a further error of ϵ, we can replace the sets U_n by a disjoint family $\{W_n\}$ covering X. Then absolute convergence is evident; and if $\sum\alpha(x_n)\neq 1$, we may choose $\epsilon>0$ so small that $\sum a(W_n)=1-d$ with $d\neq 0$.

Let $a_{ij}^*=\sum[a_{ik}\colon x_k\in W_j]$; note that $\lim_i a_{ij}^*=a(W_j)$. Let

$$c_{ij} = \frac{a_{ij}^* - a(W_j)}{d}\,.$$

Then (c_{ij}) is regular since (1) and (3) hold, and $\sum_j |c_{ij}|$ is bounded by $2/|d|$ times the bound for $\sum_j |a_{ij}|$. Since no regular matrix can sum all sequences of zeros and ones [7, p. 54], there is a subset Z of N such that $\sum_{j\in Z} c_{ij}$ does not converge, so $W=\bigcup\{W_n\colon n\in Z\}$ is an open-and-closed set for which $a(W)$ does not exist. This contradiction establishes the lemma.

COROLLARY. *For any open-and-closed set* U, $\sum[\alpha(x)\colon x\in U]=a(U)$.

PROOF. Passing to $(X\sim U)$ if necessary, we may assume that $a(U)\neq 0$. The matrix (b_{ij}) obtained by letting $b_{ij}=a_{ij}/a(U)$ if $x_j\in U$, and by letting $b_{ij}=0$ otherwise, is a regular matrix that sums each element of $C^*(U)$, so the lemma applies.

THEOREM 1. *If a regular matrix summability method* ϕ *sums all bounded continuous functions on a countably infinite topological space* $X=\{x_1, x_2, \cdots\}$, *then there is an absolutely convergent series* $\sum\alpha_n$ *with sum* 1 *such that for each* $f\in C^*(X)$, $\phi(f)=\sum\alpha_n f(x_n)$.

PROOF. The corollary shows this for characteristic functions and the rest follows from linearity and continuity.

2. **Reduction to points.** As indicated in the introduction, we can reduce the problem of which functionals $f \to \sum \alpha_n f(x_n)$ are given by regular matrices to the problem for single points, $f \to f(x)$. There is a further reduction to the case that x is the only nonisolated point. (Obviously x must be nonisolated.) We define a *heavy point* x of a countable space $\{x_1, x_2, \cdots\}$ as one such that there exists a regular matrix A such that for every bounded function f continuous at x, $\phi_A(f) = f(x)$.

THEOREM 2. *A functional $\phi(f) = \sum \alpha_n f(x_n)$ on $C^*(X)$ is representable as ϕ_A for some regular matrix A if and only if $\sum \alpha_n = 1$ and $\alpha_n = 0$ whenever x_n is not a heavy point.*

The proof will be published elsewhere, together with the results abstracted in [**4**], which tell a little about heavy points. It is easy to see that the limit of a convergent (nonconstant) sequence is a heavy point; another heavy point that is not the limit of a sequence is exhibited, essentially, in [**3**, Example 3.3].

3. **Removable points.** A point x for which every function $f \in C^*(X \sim \{x\})$ has an extension in $C^*(X)$ cannot be a heavy point; for the matrix A summing $C^*(X)$ ($\phi_A(f) = f(x)$) would, with one column deleted, sum all of $C^*(X \sim \{x\})$ (ϕ_A violating Theorem 1). As the omitted proof of Theorem 2 is long, we note that this argument works as well with $\phi_A(f) = \sum \alpha_n f(x_n)$, if $x = x_r$ has a nonzero coefficient α_r; that is, Theorem 1 suffices. Moreover, there is a trifle of extra information; if A sums every f in $C^*(X)$ to $\sum \alpha_n f(x_n)$, and $\alpha_m \neq 0$, then there is a bounded function discontinuous only at x_m that A fails to sum.

A subspace Y of a completely regular space X is said to be C^*-embedded if every $f \in C^*(Y)$ has an extension in $C^*(X)$. It is well known [**2**, p. 23] that a space X is extremally disconnected if and only if each of its dense subspaces is C^*-embedded. Thus, from Theorem 1 and the above, we have

THEOREM 3. *If the complement of each point of a countably infinite space X is C^*-embedded, in particular, if X is extremally disconnected, then no regular matrix can sum every element of $C^*(X)$.*

The complement of every point of X may be C^*-embedded without X being extremally disconnected. For example, identify two copies of a countable extremally disconnected space along a closed dense in itself subspace.

In [**6**], W. Rudin proved that if X is a countable subspace of βN, there is an $f \in C^*(X)$ that is not C_1-summable. Any such X is extremally disconnected; indeed every subspace of a countable subspace of βN is C^*-embedded [2, p. 97]. Every countable extremally disconnected space takes this form; in fact

Every extremally disconnected space X having a dense subspace of power $\mathfrak{m}$ can be embedded in βD, where D is a discrete space of power $\mathfrak{m}$.

PROOF. There is a mapping τ of D onto a dense subspace Y of X which has a continuous extension over βD onto βX [2, p. 86]. Let E be a closed subspace of βD minimal with respect to the property of being mapped onto βX by τ. Gleason shows in [1] that the restriction of τ to E is a homeomorphism since βX is extremally disconnected [2, p. 96].

This easy application of Gleason's theorem answers a question of Katětov, who asked if every extremally disconnected space, every subspace of which is normal, can be embedded in βD for some discrete space D [5].

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