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COMMUTING VECTOR FIELDS ON 2-MANIFOLDS

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We shall consider C^1 vector fields X, Y on a compact 2-manifold M . When the Lie bracket $[X, Y]$ vanishes identically on M , we say that X and Y *commute*. It was shown in [1] that every pair of commuting vector fields on the 2-sphere S^2 has a common singularity. Here we extend this result to all compact 2-manifolds with nonvanishing Euler characteristic.

Our manifolds are connected and may have boundary. The boundary of a compact 2-manifold is either empty or consists of finitely many disjoint circles. Given a C^1 vector field X on a compact manifold M , we tacitly assume that X is tangent to the boundary of M (if it exists). Then the trajectories of X are defined for all values of the parameter, and translation along them provides a (differentiable) action ξ of the additive group R on M . Given $x \in M$, one has $X(x) = 0$ if, and only if, x is a fixed point of ξ , that is, $\xi(s, x) = x$ for all $s \in R$. Let Y be another C^1 vector field on M , generating the action η of R on M . The condition $[X, Y] \equiv 0$ means that ξ and η commute, that is, $\xi(s, \eta(t, x)) = \eta(t, \xi(s, x))$ for all $x \in M$ and $s, t \in R$. Thus the pair X, Y generates an action $\phi: R^2 \times M \rightarrow M$ of the additive group R^2 on M , defined by $\phi(r, x) = \xi(s, \eta(t, x)) = \eta(t, \xi(s, x))$ for $x \in M$ and $r = (s, t) \in R^2$. Notice that $x \in M$ is a fixed point of ϕ if, and only if, x is a common singularity of X and Y , that is, $X(x) = Y(x) = 0$. These

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considerations show that our main result, Theorem B, follows from Theorem A below.

THEOREM A. *Every continuous action of the additive group R^2 on a compact 2-manifold M , with $\chi(M) \neq 0$, has a fixed point.*

THEOREM B. *Let X, Y be commuting vector fields of class C^1 on a compact 2-manifold M , with $\chi(M) \neq 0$. There exists a point $x \in M$ such that $X(x) = Y(x) = 0$.*

We sketch the proof of Theorem A.

By using the orientable double-covering, one sees that the theorem needs to be proved only for orientable 2-manifolds. For each line Z through the origin of R^2 , consider an action ζ of R on M , induced by the restriction of ϕ to Z . The action ζ has a fixed point $z \in M$, by virtue of the Lefschetz fixed point theorem, since $\chi(M) \neq 0$. The orbit of z under any 1-dimensional subgroup $W \subset R^2$, distinct from Z , is the same and coincides with its orbit $\phi(R^2 \times z)$ under the entire R^2 . The closure of $\phi(R^2 \times z)$ is compact, invariant and nonempty. So, we may choose a minimal set μ_Z , of the action ϕ , contained in that closure. This defines a map $Z \rightarrow \mu_Z$, which assigns to each line subgroup $Z \subset R^2$ a minimal set μ_Z of ϕ , which is also a minimal set for the action of each 1-dimensional subgroup $W \neq Z$ (induced by ϕ). The isotropy group of every point in μ_Z is the same and contains Z . So, if $W \neq Z$ and $x \in \mu_W \cap \mu_Z$, the isotropy group of x would contain W and Z , hence $\mu_Z = \mu_W = \{x\} = \text{fixed point under } \phi$. Suppose, by contradiction, that ϕ has no fixed point. Then, no μ_Z is a point and $\mu_Z \cap \mu_W = \emptyset$ for $W \neq Z$. Therefore, we have uncountably many (pairwise disjoint) minimal sets μ_Z of ϕ . Fix a line subgroup $Z_0 \subset R^2$. All the sets μ_Z with $Z \neq Z_0$ are minimal under the action ζ_0 of R on M , induced by the restriction of ϕ to Z_0 . Now, a compact 2-manifold may contain only finitely many nontrivial (this means not points or circles) minimal sets under a given continuous action of the reals. (I am indebted to Mauricio Peixoto for this crucial remark.) Hence, there exist lines Z_1, \dots, Z_k , through the origin of R^2 , such that, except for $\mu_{Z_0}, \mu_{Z_1}, \dots, \mu_{Z_k}$, all the other minimal sets μ_Z are circles. This means that they are actually closed orbits of the action ϕ , as well as of every action ζ induced by ϕ on any line subgroup $Z \in \{Z_0, Z_1, \dots, Z_k\}$. We now wish to cut successively the manifold M along these circles, thus reducing its genus g , and to prove the theorem by induction on g . In order to do this, we first observe that no closed orbit μ may bound a disc D in M , since D would then be invariant under ϕ , while every action of R^2 on a disc has a fixed point

(as follows from [1]) and we are admitting no fixed points. So, when M is cut along a closed orbit μ , its genus is decreased, unless μ bounds a cylinder together with some boundary circle Γ of M . Therefore, we must show that μ may be chosen in such a way that it bounds no cylinder together with any boundary circle of M . This is done as follows: consider the collection $\{C_\alpha\}$ of all cylinders bounded by a fixed boundary circle Γ of M and a closed 1-dimensional orbit μ_α of ϕ . For convenience, we let $\Gamma \subset C_\alpha$ but $C_\alpha \cap \mu_\alpha = \emptyset$, so each C_α is open in M . The collection $\{C_\alpha\}$ is seen to be linearly ordered by inclusion. Hence, its union $C = \cup C_\alpha$ is still a cylinder, open in M . Moreover, the point-set boundary of C in M is a closed 1-dimensional orbit ν of ϕ . Therefore, its closure \bar{C} is a closed cylinder in M . Notice that all boundary circles of M are invariant under ϕ so, unless we admit fixed points, they are closed orbits of ϕ . Hence, \bar{C} is disjoint from all boundary circles of M , except Γ . It follows that $M' = M - C$ is a compact 2-manifold, homeomorphic to M , with ν replacing Γ as a boundary circle. M' is invariant under ϕ and no closed 1-dimensional orbit of ϕ in M' bounds a cylinder together with ν . Repeating this procedure on the other boundary circles of M , we obtain finally a compact 2-manifold $M_0 \subset M$, homeomorphic to M (in particular $\chi(M_0) \neq 0$) invariant under ϕ and, except for the boundary circles of M_0 , no closed 1-dimensional orbit of ϕ in M_0 bounds a cylinder together with a boundary circle of M . There are uncountably many 1-dimensional closed orbits of ϕ in M_0 . Choose one of them, say μ , that is not a boundary circle of M_0 . Cutting M along μ , we either obtain one or two compact 2-manifolds of lower genus than M . The induction process is then settled. The initial step, for manifolds of genus zero, is treated much in the same way, with another induction, this time on the number of boundary curves, and the problem is eventually reduced to showing that every continuous action of R^2 on a 2-disc has a fixed point. Putting two copies of the disc together and gluing them along the boundary, we see that this follows from the existence of fixed points for actions of the plane on S^2 .

REMARK (*Added in proof*). Replacing each line Z by a hyperplane Z in the above argument, one obtains an inductive proof that the theorem holds not only for R^2 but for an arbitrary R^n , as well.

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