QUASICONFORMAL MAPPINGS IN SPACE

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1. **Introduction.** Suppose that $w(z)$ is a homeomorphism of a plane domain D onto a plane domain D'. We introduce, for convenience, the following functions which measure how much infinitesimal circles and areas are distorted under the mapping $w(z)$ at each point z_0 in D:

$$(1) \qquad H(z_0) = \limsup_{r\to 0} \frac{L(z_0, r)}{l(z_0, r)}, \qquad J(z_0) = \limsup_{r\to 0} \frac{m(U')}{m(U)}.$$

Here

$$L(z_0, r) = \sup_{|z-z_0|=r} |w(z) - w(z_0)|, \quad l(z_0, r) = \inf_{|z-z_0|=r} |w(z) - w(z_0)|,$$

U' denotes the image of U, the disk $|z-z_0|<r$, and m denotes Lebesgue plane measure. If $w(z)$ is differentiable at z_0, then $w(z)$ is locally affine at z_0 and maps the infinitesimal circles $|z-z_0|=\epsilon$ onto infinitesimal ellipses; $H(z_0)$ gives the ratio of the major to minor axes and $J(z_0)$ is the absolute value of the Jacobian.

Suppose next that $w(z)$ is continuously differentiable with $J(z)>0$ everywhere in D. Then $w(z)$ is said to be a *K-quasiconformal mapping*, in the classical sense, if $w(z)$ satisfies the dilatation condition

$$(2) \qquad H(z) \leqq K, \qquad 1 \leqq K < \infty,$$

everywhere in D. A homeomorphism is said to be quasiconformal if it is K-quasiconformal for some K. If $w(z)$ is sense-preserving with $H(z)=1$ in D, then the real and imaginary parts of $w(z)$ satisfy the Cauchy-Riemann equations and $w(z)$ is an analytic function of z. Hence a sense-preserving 1-quasiconformal mapping is conformal in the ordinary sense.

Quasiconformal mappings arise very naturally in complex function theory, for example in the study of multiply connected domains and in the Teichmüller problem. They are encountered also in the theory of partial differential equations as the univalent solutions of Beltrami systems. Finally, the study of such mappings is interesting in its own right, for though the theory usually parallels that of conformal map-

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ping, there are striking instances where the analogy breaks down. Moreover, this study sometimes casts new light on the theory of conformal mapping, since often one must employ different methods when dealing with this more general class of mappings.

Obviously one can consider an analogous class of homeomorphisms in Euclidean 3-space and we shall discuss here a few of the results which have been established for such mappings. However in order to motivate what follows, we shall illustrate first how one can obtain certain properties of plane quasiconformal mappings by means of the moduli of rings.

2. **Plane rings.** A *ring* R is defined to be a finite doubly connected domain, that is, a domain whose complement with respect to the extended plane consists of two components. We denote these components by C_0 and C_1, and we assume that C_1 contains the point at infinity. Each ring R can be mapped conformally onto an annulus $a<|w|<b$ so that the circles $|w|=a$ and $|w|=b$ correspond to ∂C_0 and ∂C_1, respectively. Then $0<a<b<\infty$ if and only if C_0 and C_1 are nondegenerate. The conformal invariant

$$\text{mod}\, R = \log \frac{b}{a} \tag{3}$$

is called the *modulus* of R.

There are other ways to define mod R. For example if C_0 and C_1 are nondegenerate and if $w(z)$ is the above mentioned conformal mapping of R onto $a<|w|<b$, then

$$v(z) = \frac{\log |w(z)/a|}{\log b/a} \tag{4}$$

is harmonic in R and has boundary values 0 on ∂C_0 and 1 on ∂C_1. Next an elementary calculation shows that the modulus can be expressed in terms of the capacity $C(R)$ of the ring as follows:

$$C(R) = \int_R |\nabla v|^2 d\sigma = \frac{2\pi}{\text{mod}\, R}\,. \tag{5}$$

Finally if we appeal to the Dirichlet principle we obtain

$$\frac{2\pi}{\text{mod}\, R} = C(R) = \inf_u \int_R |\nabla u|^2 d\sigma, \tag{6}$$

where the infimum is taken over all functions $u=u(z)$ which are continuously differentiable in R with boundary values 0 on ∂C_0 and 1 on ∂C_1. It is easy to see that (6) still holds when C_0 or C_1 reduces to a point and hence we obtain an alternative definition for the modulus which does not depend on conformal mapping. This is important since it suggests how to define the modulus of a ring in space where the only conformal mappings are the Moebius transformations.

One can also define the capacity, and hence the modulus, of a ring R by means of extremal lengths. For each non-negative Borel measurable function $f=f(z)$ defined in R we let

$$L_1(f) = \inf_{\gamma_1} \int_{\gamma_1} f ds, \qquad L_2(f) = \inf_{\gamma_2} \int_{\gamma_2} f ds, \tag{7}$$

where γ_1 denotes any locally rectifiable curve in R which joins the boundary components ∂C_0 and ∂C_1, and γ_2 any locally rectifiable curve in R which separates ∂C_0 and ∂C_1. Next set

$$A(f) = \int_R f^2 d\sigma. \tag{8}$$

Then it is not difficult to show [2] that

$$\sup_f \frac{L_2(f)^2}{A(f)} = C(R) = \inf_f \frac{A(f)}{L_1(f)^2}, \tag{9}$$

where the supremum is taken over all f for which $L_2(f)$ and $A(f)$ are not simultaneously 0 or ∞, and the infimum over all f for which $A(f)$ and $L_1(f)$ are not simultaneously 0 or ∞.

3. **Extremal rings.** There are a pair of extremal rings, first studied by Grötzsch [12] and Teichmüller [25], which play an important role in the distortion theory of quasiconformal mappings.

Given $a>1$ and $b>0$, we denote by $R_G=R_G(a)$ the ring bounded by the disk $|z|\leqq 1$ and the ray $a\leqq x\leqq\infty$, $y=0$, and we denote by $R_T=R_T(b)$ the ring bounded by the segment $-1\leqq x\leqq 0$, $y=0$ and the ray $b\leqq x\leqq\infty$, $y=0$. Then if we set

$$\operatorname{mod} R_G = \log \Phi_2(a), \qquad \operatorname{mod} R_T = \log \Psi_2(b), \tag{10}$$

it is easy to verify [25] that $\Phi_2(a)/a$ is nondecreasing in a, that

$$\lim_{a\to\infty} \frac{\Phi_2(a)}{a} = 4, \tag{11}$$

and that

$$\Psi_2(b) = \Phi_2((b+1)^{1/2})^2. \tag{12}$$

The rings R_G and R_T have the following extremal property [25]. Suppose that R is a ring and that z_0 is a point of C_0. If C_0 contains all points at distance a from z_0 and if C_1 contains at least one point at distance b from z_0, then

$$\text{mod } R \leqq \text{mod } R_G\left(\frac{b}{a}\right). \tag{13}$$

If C_0 contains at least one point at distance a from z_0 and if C_1 contains at least one point at distance b from z_0, then

$$\text{mod } R \leqq \text{mod } R_T\left(\frac{b}{a}\right). \tag{14}$$

4. **Modulus of a ring under a quasiconformal mapping.** Now suppose that $w(z)$ is a K-quasiconformal mapping of a domain D onto D' and that R is a ring with closure $\overline{R}\subset D$. Then $w(z)$ carries R onto a ring $R'\subset D'$. Next suppose that $v=v(w)$ is continuously differentiable in R' with boundary values 0 on $\partial C_0'$ and 1 on $\partial C_1'$, and set $u(z)=v(w(z))$. Then either $u(z)$ or $1-u(z)$ has the same properties with respect to R and

$$|\nabla u(z)|^2 \leqq H(z)J(z)\,|\nabla v(w)|^2 \leqq KJ(z)\,|\nabla v(w)|^2$$

at each point of R. Hence with (6) we have

$$C(R) \leqq \int_R |\nabla u|^2 d\sigma \leqq K\int_R J\,|\nabla v|^2 d\sigma = K\int_{R'} |\nabla v|^2 d\sigma,$$

and taking the infimum over all such functions v yields $C(R)\leqq KC(R')$ or mod $R'\leqq K$ mod R. Since the inverse mapping $z(w)$ is also K-quasiconformal, we conclude that

$$\frac{1}{K}\text{mod } R \leqq \text{mod } R' \leqq K \text{ mod } R \tag{15}$$

for all rings R with $\overline{R}\subset D$.

5. **An application.** We illustrate how one can use the results of §§3 and 4 to obtain some properties of quasiconformal mappings.

Suppose that $w(z)$ is a K-quasiconformal mapping of a domain D onto D', that z_0 is a point of D, and that d and d' denote the distances from z_0 to ∂D and $w(z_0)$ to $\partial D'$, respectively. Next for $0<a<b<d$ let R denote the annulus $a<|z-z_0|<b$, let R' denote the image of R under $w(z)$, and set

$$a' = \min_{|z-z_0|=a} |w(z) - w(z_0)| > 0, \quad b' = \min_{|z-z_0|=b} |w(z) - w(z_0)| < d'.$$

It is easy to see that C_0' contains all points within distance a' of

$w(z_0)$ and that C_1' contains at least one point at distance b' from $w(z_0)$. Hence we have

$$\log \frac{b}{a} = \operatorname{mod} R \leqq K \operatorname{mod} R' \leqq K \log 4 \frac{b'}{a'},$$

or simply

$$\left(\frac{b}{a}\right)^{1/K} \leqq 4 \frac{b'}{a'} < 4 \frac{d'}{a'}, \tag{16}$$

from (10), (11), (13) and (15).

If we hold a fixed and let $b \to d$, then (16) yields

$$\frac{a'}{a^{1/K}} \leqq 4 \frac{d'}{d^{1/K}}.$$

In particular it follows that $d' = \infty$ whenever $d = \infty$, and hence that D' is the whole plane whenever D is. Next if $d < \infty$ and if we let $a \to 0$, we obtain

$$\liminf_{z \to z_0} \frac{|w(z) - w(z_0)|}{|z - z_0|^{1/K}} \leqq 4 \frac{d'}{d^{1/K}}. \tag{17}$$

In the special case where $w(z)$ is a conformal mapping, $K = 1$ and (17) give us the Koebe Viertelsatz $|w'(z_0)| \leqq 4d'/d$.

6. **Space rings.** We now turn to the study of quasiconformal mappings in space. The main problem is to obtain global properties for these mappings using only the fact that they satisfy a local dilatation condition similar to (2). We have shown in §§4 and 5 how one can do this for plane quasiconformal mappings using the moduli of plane rings, and so we begin by introducing an analogous modulus for space rings.

A *space ring* R is defined to be a finite domain whose complement in the Moebius space consists of two components. We denote these components by C_0 and C_1, and let C_1 contain the point at infinity. Next following Loewner [**14**], we define the *conformal capacity* of a ring R as follows:

$$\Gamma(R) = \inf_u \int_R |\nabla u|^3 d\omega, \tag{18}$$

where the infimum is taken over all functions $u = u(x)$, $x = (x_1, x_2, x_3)$, which are continuously differentiable in R and have boundary values 0 on ∂C_0 and 1 on ∂C_1. We then define the *modulus* of R by means of the relation

(19) $$\operatorname{mod} R = \left(\frac{4\pi}{\Gamma(R)}\right)^{1/2}.$$

These two equations are the space analogues of (6). In particular if R is the spherical annulus bounded by concentric spheres of radii a and b, $a<b$, it is easy to show [8] that

(20) $$\operatorname{mod} R = \log \frac{b}{a}\cdot$$

When R is a plane ring with nondegenerate boundary components, then the harmonic function $v(z)$, given in (4), is the unique solution for the extremal problem in (6). Similarly, if R is a space ring which has nondegenerate boundary components, there exists a unique function $v=v(x)$ with the following properties: v is continuous and ACL in R, v has boundary values 0 on ∂C_0 and 1 on ∂C_1, v is differentiable a.e. in R, and

(21) $$\Gamma(R) = \int_R |\nabla v|^3 d\omega.$$[2]

We call v the *extremal function* for the ring R. It satisfies the variational condition

(22) $$\int_R |\nabla v| \nabla v \cdot \nabla w d\omega = 0$$

for each function $w=w(x)$ which is continuously differentiable and has compact support in R. (See [10].)

In the case of a plane ring, the function $v(z)$ in (4) has a nonvanishing gradient and $|\nabla v(z)|$ is bounded away from 0 and ∞ on each compact set in the ring. The situation in space is different, partly because a space ring need not be topologically equivalent to a spherical annulus. For example, if $v(x)$ is the extremal function for the ring bounded by the circle $x_1^2+x_2^2=1$, $x_3=0$ and by the spherical surface $|x|=2$, then it is clear on the basis of symmetry that $|\nabla v(x)|$ must vanish at the origin if $v(x)$ is differentiable there.

It is not yet known how smooth the extremal function $v(x)$ is when R is an arbitrary space ring with nondegenerate boundary components. However if $|\nabla v(x)|$ is bounded away from 0 and ∞ a.e. on each compact set in R, then one can use (22) to show [10] that

[2] A function is said to be *absolutely continuous on lines* or ACL in a domain if, for each sphere U with closure in the domain, the function is absolutely continuous on almost all line segments in U which are parallel to the coordinate axes.

$v(x)$ is real analytic in R and satisfies the quasilinear elliptic differential equation

$$\operatorname{div}(|\nabla v| \nabla v) = 0. \tag{23}$$

When R satisfies certain rather restrictive geometrical conditions, it is possible to establish the above a priori bounds for $|\nabla v(x)|$, and hence show that $v(x)$ is real analytic and satisfies (23).

The extremal function for a spherical annulus is a linear function of $\log r$, where r denotes the distance from the center of the annulus to the point x.

7. **Extremal lengths.** According to a well-known theorem due to Liouville, the Moebius transformations are the only conformal mappings in space. Thus a space ring R can be mapped conformally onto a spherical annulus only if it is bounded by two spheres or a sphere and a plane, and there is no space analogue for the first definition we gave for the modulus of a plane ring. However it is interesting to note that there are space analogues for the extremal length definitions given in (9).

For each non-negative Borel measurable function $f=f(x)$ defined in a space ring R we let

$$L(f) = \inf_{\gamma} \int_{\gamma} f ds, \qquad A(f) = \inf_{\Sigma} \int_{\Sigma} f^2 d\sigma, \tag{24}$$

where γ denotes any locally rectifiable curve in R which joins ∂C_0 and ∂C_1, and Σ any piecewise smooth compact surface in R which separates ∂C_0 and ∂C_1. Next set

$$V(f) = \int_R f^3 d\omega. \tag{25}$$

Then we can show [9] that

$$\sup_f \frac{A(f)^3}{V(f)^2} = \Gamma(R) = \inf_f \frac{V(f)}{L(f)^3}, \tag{26}$$

where the supremum is taken over all f for which $A(f)$ and $V(f)$ are not simultaneously 0 or ∞, and the infimum over all f for which $V(f)$ and $L(f)$ are not simultaneously 0 or ∞.

8. **Extremal rings in space.** Given $a>1$ and $b>0$, we let $R_G=R_G(a)$ denote the space ring bounded by the sphere $|x| \leqq 1$ and the ray $a \leqq x_1 \leqq \infty$, $x_2=x_3=0$, and $R_T=R_T(b)$ the ring bounded by the segment $-1 \leqq x_1 \leqq 0$, $x_2=x_3=0$ and the ray $b \leqq x_1 \leqq \infty$, $x_2=x_3=0$.

These are the space analogues of the Grötzsch and Teichmüller rings considered in §3. If as in (10) we set

$$\text{mod } R_G = \log \Phi_3(a), \qquad \text{mod } R_T = \log \Psi_3(b), \tag{27}$$

then it is easy to show that $\Phi_3(a)/a$ is nondecreasing in a, that

$$\lim_{a\to\infty} \frac{\Phi_3(a)}{a} = \lambda \quad \text{where} \quad 4 \leqq \lambda < 12.4 \cdots, \tag{28}$$

and that

$$\Psi_3(b) = \Phi_3((b+1)^{1/2})^2. \tag{29}$$

We can further show that the moduli of the Grötzsch and Teichmüller rings in space are not less than the moduli of the corresponding plane rings, that is

$$\Phi_2(a) \leqq \Phi_3(a) \quad \text{and} \quad \Psi_2(b) \leqq \Psi_3(b). \tag{30}$$

(See [8; 9].)

As in the plane case, the importance of the rings R_G and R_T depends upon the fact that they have the following extremal properties [8]. Suppose that R is a space ring and that P is a point of C_0. If C_0 contains all points at distance a from P and if C_1 contains at least one point at distance b from P, then

$$\text{mod } R \leqq \text{mod } R_G \left(\frac{b}{a}\right).^{8} \tag{31}$$

If C_0 contains at least one point at distance a from P and if C_1 contains at least one point at distance b from P, then

$$\text{mod } R \leqq \text{mod } R_T \left(\frac{b}{a}\right). \tag{32}$$

9. **Quasiconformal mappings in space.** We begin with the study of quasiconformal mappings in space. Suppose that $y(x)$ is a homeomorphism of D onto D', where D and D' are finite domains in Euclidean 3-space. As in §1 we define some functions to measure local distortion under $y(x)$ at each point P in D:

$$\begin{aligned} H(P) &= \limsup_{r\to 0} \frac{L(P,r)}{l(P,r)}, \qquad I(P) = \limsup_{r\to 0} \frac{L(P,r)}{r}, \\ J(P) &= \limsup_{r\to 0} \frac{m(U')}{m(U)}. \end{aligned} \tag{33}$$

[8] B. V. Šabat established this inequality in [24], using extremal lengths and a slightly different definition for mod R.

Here

$$L(P,r) = \sup_{|x-P|=r} |y(x) - y(P)|, \quad l(P,r) = \inf_{|x-P|=r} |y(x) - y(P)|,$$

U' denotes the image of U, the sphere $|x-P|<r$, and m denotes Lebesgue measure in space.

The homeomorphism $y(x)$ is said to be a *K-quasiconformal mapping*, in the classical sense, if $y(x)$ is continuously differentiable with $J(x)>0$ and $H(x)\leqq K$ everywhere in D. An elementary adaptation of the argument in §4 then shows that, if R is a space ring with $\overline{R}\subset D$,

$$\Gamma(R) \leqq K^2\Gamma(R') \quad \text{or} \quad \operatorname{mod} R' \leqq K \operatorname{mod} R,$$

where R' is the image of R under $y(x)$. Since the inverse mapping is K-quasiconformal, we conclude that

$$\frac{1}{K} \operatorname{mod} R \leqq \operatorname{mod} R' \leqq K \operatorname{mod} R \tag{34}$$

for all space rings R with $\overline{R}\subset D$.

Now all of the most important geometric properties for classical K-quasiconformal mappings can be derived from the fact that $y(x)$ is a homeomorphism which satisfies the inequality (34). Hence we are led to adopt the following slightly more general definition for quasiconformal mappings.

DEFINITION 1. *A homeomorphism $y(x)$ of a domain D is said to be a K-quasiconformal mapping, $1\leqq K<\infty$, if the inequality* (34) *holds for each bounded ring R with $\overline{R}\subset D$. A quasiconformal mapping is one which is K-quasiconformal for some K.*[4]

From the preceding discussion it is clear that a mapping which is K-quasiconformal in the classical sense is also K-quasiconformal according to Definition 1.

This definition for quasiconformality can be defended on other than purely aesthetic grounds. For example, consider the homeomorphism

$$y_1 = x_1, \qquad y_2 = x_2, \qquad y_3 = f(x_3), \tag{35}$$

where $f(t)$ is continuously differentiable in $0<t<\infty$ with

$$\lim_{t\to 0+} f(t) = 0, \qquad \frac{1}{K} \leqq f'(t) \leqq K.$$

[4] J. Väisälä has used extremal lengths to define and study a class of quasiconformal mappings in [26; 27]. A mapping is K-quasiconformal by Väisälä's definition if and only if it is $K^{1/2}$-quasiconformal by Definition 1. B. V. Šabat has also used extremal lengths to derive a number of interesting properties of quasiconformal mappings in [23; 24].

Then $y(x)$ is a classical K-quasiconformal mapping of the half space $x_3>0$ onto the half space $y_3>0$, and we can reflect in the boundary planes to obtain a homeomorphism of the x-space onto the y-space. But the extended mapping will not, in general, be continuously differentiable in the boundary planes. Hence we conclude that the reflection principle does not hold for mappings which are K-quasiconformal in the classical sense.

It is also easy to show, by means of an example, that a homeomorphism, which is the uniform limit of classical K-quasiconformal mappings, need not itself be continuously differentiable. Thus the usual compactness result does not hold for this class of mappings.

The above reflection and compactness principles fail for classical K-quasiconformal mappings simply because of the a priori differentiability hypothesis in the definition. The example in (35) further shows that there is still difficulty, even if we allow the existence of an exceptional set of isolated points where the mapping may fail to be differentiable. On the other hand, there are no differentiability hypotheses in Definition 1, and we shall see that the reflection principle and the usual compactness theorems are valid for our slightly broader class of mappings.

Finally we should observe that the above remarks apply equally well to the classical plane quasiconformal mappings $w(z)$ defined in §1. For this reason, many authors have begun to study a slightly more general class of plane quasiconformal mappings, namely those plane homeomorphisms which do not change the moduli of quadrilaterals by more than some fixed factor. (See, for example, [**1**; **3**; **17**; **20**].) This class can also be defined in terms of rings as follows [**11**].

DEFINITION 2. *A homeomorphism $w(z)$ of a plane domain D is said to be a K-quasiconformal mapping, $1 \leqq K < \infty$, if the inequality*

$$\text{mod } R' \leqq K \text{ mod } R \tag{36}$$

holds for each bounded plane ring R with $\overline{R} \subset D$, where R' denotes the image of R under $w(z)$.

Now this means that any homeomorphism $w(z)$ which satisfies the inequality (36) must also satisfy the double inequality

$$\frac{1}{K} \text{ mod } R \leqq \text{mod } R' \leqq K \text{ mod } R.$$

Hence we see that the class of K-quasiconformal mappings considered in Definition 1 is the space analogue for the very widely studied class of plane mappings of Definition 2.

10. The 1-quasiconformal mappings. Since the modulus condition (34) is symmetric, we see that the inverse of a K-quasiconformal mapping is also K-quasiconformal. It is furthermore clear that the composition of two mappings which are K_1- and K_2-quasiconformal is a K_1K_2-quasiconformal mapping. In particular the composition of a K-quasiconformal mapping with a 1-quasiconformal mapping is again K-quasiconformal.

It is therefore important to identify the 1-quasiconformal mappings in space. Suppose that $y(x)$ is a homeomorphism which is the restriction of a Moebius transformation to a domain D. Then $y(x)$ is real analytic with $J(x)>0$ and $H(x)=1$ everywhere in D,

$$\text{mod } R' = \text{mod } R \tag{37}$$

for each space ring R with $\overline{R}\subset D$, and hence $y(x)$ is a 1-quasiconformal mapping. Conversely, suppose that $y(x)$ is a homeomorphism of a domain D which satisfies (37) for all bounded rings R with $\overline{R}\subset D$. Then it follows that $y(x)$ must preserve the class of extremal functions for rings. Next by appealing to the analyticity result in §6 and the distortion theorem in §13, we can show that $y(x)$ is real analytic with $J(x)>0$ and $H(x)=1$ everywhere in D, and we conclude from the classical theorem of Liouville that $y(x)$ is the restriction of a Moebius transformation to D. We thus obtain the following result [**10**].

THEOREM 1. *A homeomorphism $y(x)$ of a domain D is 1-quasiconformal if and only if it is the restriction of a Moebius transformation to D.*

It is also of interest to observe that we can use Theorem 1 to establish a very general form of the above mentioned theorem of Liouville [**10**]. (See also [**22**].)

THEOREM 2. *If $y(x)$ is a homeomorphism of a domain D, if $H(x)<\infty$ everywhere in D, and if $H(x)=1$ a.e. in D, then $y(x)$ is the restriction of a Moebius transformation to D.*

Theorem 2 is the space analogue of a well-known theorem due to Menchoff [**15**].

11. Analytic properties of K-quasiconformal mappings. Suppose that $y(x)$ is a K-quasiconformal mapping of D. In the special case where $K=1$, we see from above that $y(x)$ is real analytic and that $J(x)>0$ and $H(x)=1$ everywhere in D. We consider next what can be said in the general case where $K>1$.

First the fact that $y(x)$ satisfies the one-sided modulus inequality

$$\text{mod } R' \leqq K \text{ mod } R \tag{38}$$

implies that

$$H(x) < e^{6K} \tag{39}$$

everywhere in D; the proof makes use of the extremal property of the ring R_T described in §8. Next we can show that inequality (39) implies that $y(x)$ is ACL in D and that $y(x)$ is differentiable a.e. in D. It is then easy to prove from (38) that

$$I(x)^3 \leqq K^2 J(x) \tag{40}$$

at each point where $y(x)$ has a differential, and hence (40) holds a.e. in D. The above analytic properties, and what they imply, are all we can expect to establish for a homeomorphism $y(x)$ satisfying the inequality (38). For we can adapt the argument of §4 to reverse the above steps and obtain the following result [**10**].

THEOREM 3. *A homeomorphism $y(x)$ of D satisfies the inequality* (38) *for all bounded rings R with $\overline{R} \subset D$ if and only if $y(x)$ is* ACL *in D and satisfies the inequality* (40) *a.e. in D.*

In particular, if we apply Theorem 3 to the inverse mapping $x(y)$ and let $I^*(y)$ and $J^*(y)$ denote the distortion functions for $x(y)$ corresponding to $I(x)$ and $J(x)$, we obtain an analytic characterization for quasiconformal mappings [**10**]. (See also [**26**].)

THEOREM 4. *A homeomorphism $y(x)$ of D onto D' is K-quasiconformal if and only if $y(x)$ and $x(y)$ are* ACL *with*

$$I(x)^3 \leqq K^2 J(x) \quad \text{and} \quad I^*(y)^3 \leqq K^2 J^*(y) \tag{41}$$

a.e. in D and D', respectively.

An unfortunate feature of this characterization is that it involves both the mapping and its inverse. This is partly due to the fact that we have employed the two-sided modulus inequality (34) in Definition 1. We observed in §9 that it was sufficient to use the one-sided modulus inequality (36) in Definition 2 for plane mappings. That is, a plane mapping which satisfies a one-sided inequality automatically satisfies the two-sided inequality. The following theorem shows that this is almost true in space. (See [**10**; **26**].)

THEOREM 5. *If $y(x)$ is a homeomorphism of D which satisfies* (38) *for all bounded rings R with $\overline{R} \subset D$, then $y(x)$ is a K^2-quasiconformal mapping. The bound K^2 is best possible.*

Hence in space, a mapping which satisfies a one-sided inequality with K satisfies the two-sided inequality with K^2. The mapping

$$y_1 = x_1, \qquad y_2 = K^2 x_2, \qquad y_3 = K^2 x_3$$

shows that the bound K^2 cannot be improved.

Finally we have the following result which shows that a homeomorphism which is quasiconformal is a measurable mapping. (See [**10**; **26**].)

THEOREM 6. *If $y(x)$ is a quasiconformal mapping of D, then $J(x) > 0$ a.e. in D and $y(x)$ maps each measurable set $E \subset D$ onto a measurable set E' with*

$$m(E') = \int_E J d\omega. \tag{42}$$

12. **Boundary correspondence.** If D and D' are plane Jordan domains and if $w(z)$ is a conformal mapping of D onto D', then a well-known theorem of Carathéodory asserts that $w(z)$ can be extended to be a homeomorphism of $\overline{D}$ onto $\overline{D}'$. The conclusion remains valid when $w(z)$ is a plane quasiconformal mapping. To show this one first establishes the result for the case where D and D' are open disks. The general case can then be reduced to this case by means of the Riemann mapping theorem and the above mentioned theorem of Carathéodory.

The situation in space is much more complicated, due to the fact that there is no analogue of the Riemann mapping theorem. Nevertheless one can prove the following result [**27**]. (For the special case where D' is a sphere, see [**6**; **10**].)

THEOREM 7. *If $y(x)$ is a quasiconformal mapping of the unit sphere D onto a bounded domain D' and if D' is locally connected at each point of its boundary, then $y(x)$ can be extended to be a homeomorphism of $\overline{D}$ onto $\overline{D}'$.*

In particular if $y(x)$ is a K-quasiconformal mapping of $x_3 > 0$ onto $y_3 > 0$, one can apply this result to show that $y(x)$ can be extended to be a homeomorphism of $x_3 \geqq 0$ onto $y_3 \geqq 0$. Reflection in the boundary planes $x_3 = 0$ and $y_3 = 0$ then yields a homeomorphism of the x-space onto the y-space, and we can show that the mapping so obtained is still K-quasiconformal. If we then compare the distortion function $H(x)$ for $y(x)$ with the corresponding function for the boundary mapping, we obtain the following result [**10**].

THEOREM 8. *If $y(x)$ is a quasiconformal mapping of $x_3 > 0$ onto $y_3 > 0$, then $y(x)$ can be extended to be a homeomorphism of $x_3 \geqq 0$ onto $y_3 \geqq 0$, and the induced boundary correspondence is itself a plane quasiconformal mapping of $x_3 = 0$ onto $y_3 = 0$.*

We note that this boundary correspondence is absolutely continuous or measurable as a plane mapping, that is it maps plane measurable sets onto plane measurable sets. (See [**3**; **11**; **18**].) An example due to Beurling and Ahlfors [**5**] shows that this need not be true of the boundary correspondence induced by a quasiconformal mapping of a half plane onto a half plane.

13. **General distortion theorem.** If $w(z)$ is a plane K-quasiconformal mapping, then $w(z)$ satisfies a Hölder condition of order $1/K$ at each point. The space analogue of this result, as well as a number of other important results, can be obtained from the following distortion theorem [**10**].

THEOREM 9. *For each K, $1 \leqq K < \infty$, there exists a distortion function $\Theta_K(t)$ with the following properties. $\Theta_K(t)$ is increasing and continuous for $0<t<1$ and*

$$\lim_{t\to 0+} \frac{\Theta_K(t)}{t^{1/K}} = \lambda^{2-1/K}, \qquad \lim_{t\to 1-} \Theta_K(t) = \infty, \tag{43}$$

where λ is the constant defined in (28). *Next, if $y(x)$ is a K-quasiconformal mapping of D onto D' and if D has a finite boundary point, then D' has a finite boundary point and*

$$\frac{|y(P) - y(Q)|}{d'} \leqq \Theta_K\left(\frac{|P-Q|}{d}\right) \tag{44}$$

for each pair of points P and Q in D with $|P-Q| < d$, where d and d' denote the distances from P to ∂D and $y(P)$ to $\partial D'$, respectively.

The proof for this result is very similar to the argument given in §5. It uses only the extremal properties of the rings R_G and R_T, described in §8, and the fact that $y(x)$ satisfies the modulus inequality

$$\frac{1}{K} \operatorname{mod} R \leqq \operatorname{mod} R'$$

for each bounded ring R with $\bar{R} \subset D$.

With (43) and (44) it is easy to show that if $y(x)$ is a K-quasiconformal mapping of D, then for each compact set $E \subset D$ there exists a finite constant A such that

$$|y(P) - y(Q)| \leqq A\,|P - Q|^{1/K} \tag{45}$$

for all P and Q in E. (See [**6**; **10**; **21**].)

In particular if $y(x)$ maps $|x| < 1$ K-quasiconformally onto $|y| < 1$ so that $y(0) = 0$, we can show that (45) holds for all P and Q in $|x| < 1$

with A an absolute constant, $A < 620$. (See [**10**; **24**].) This is the space form of a theorem for plane quasiconformal mappings due to Ahlfors [**1**] and Mori [**16**].

14. Compactness theorems. As we noted in §9, one of the advantages of our definition for quasiconformality over the classical one is that the quasiconformal mappings of Definition 1 have the following compactness property [**10**].

THEOREM 10. *If a sequence of K-quasiconformal mappings $\{y_n(x)\}$ converges uniformly on each compact set in D to a homeomorphism $y(x)$, then $y(x)$ is a K-quasiconformal mapping.*

If we combine this result with Theorem 9, we can establish the following theorem on normal families [**10**].

THEOREM 11. *If P and P' are fixed points in D and D' and if D has a finite boundary point, then the K-quasiconformal mappings of D onto D', which map P onto P', form a closed normal family.*

Now let $y(x)$ be an arbitrary homeomorphism of the space, let $SyT(x)$ denote the composition of $y(x)$ with any pair of similarity mappings $S(y)$ and $T(x)$, and let P and Q denote a pair of distinct fixed points. We say that a family of homeomorphisms of the space satisfies the *condition* (A) if each sequence of homeomorphisms, which map P and Q onto themselves, contains a subsequence which converges uniformly on compact sets to a homeomorphism.

We then have the following compactness criterion for quasiconformality [**10**]. (See also [**5**].)

THEOREM 12. *The homeomorphism $y(x)$ is a quasiconformal mapping if and only if the family of all mappings $SyT(x)$ satisfies the condition* (A).

Theorem 11 implies that the K-quasiconformal mappings of the space, which map P and Q onto themselves, form a closed normal family. Hence if $y(x)$ is quasiconformal, the family of mappings $SyT(x)$ obviously satisfies the condition (A). The converse result follows from the fact that a homeomorphism $y(x)$ is quasiconformal if and only if $H(x)$ is bounded.

We remark, in conclusion, that one also can use Theorems 9 and 10 to establish space analogues of the theorems of Carathéodory on the conformal mappings of variable domains.

15. Coefficient of quasiconformality. One of the most important results in complex analysis is the Riemann mapping theorem. It asserts that every simply-connected plane domain D, which has at least one finite boundary point, can be mapped conformally onto the unit disk. Hence in the case of the plane, there are many different domains D which are 1-quasiconformally equivalent to the unit disk.

Theorem 1 shows us that the situation is quite changed in space. That is, a space domain D can be mapped 1-quasiconformally onto the unit sphere if and only if D is a sphere or a half space. It is therefore natural to ask if there exist other space domains D which can be mapped K-quasiconformally onto the unit sphere with K arbitrarily close to 1.

We can reformulate this question as follows. For each domain D let $K(D)$ denote the infimum of the numbers K for which there exists a K-quasiconformal mapping $y(x)$ of D onto $|y|<1$; if no such mapping exists, set $K(D)=\infty$. We call $K(D)$ the *coefficient of quasiconformality* for D. The above question asks us to identify the domains for which $K(D)=1$.

Theorems 10 and 11 imply the following result on the existence of extremal quasiconformal mappings for domains with a finite coefficient of quasiconformality [**10**].

THEOREM 13. *If $K(D)<\infty$, there exists a $K(D)$-quasiconformal mapping $y(x)$ of D onto $|y|<1$.*

We conclude from Theorems 1 and 13 that $K(D)=1$ if and only if D is a sphere or half space. In other words, $K(D)>1$ for essentially all domains D in space, and one is now led to ask how $K(D)$ depends upon the geometrical properties of D.

We observe first that Theorem 7 can be inverted to yield the following interesting result [**27**].

THEOREM 14. *If D is a bounded domain which is locally connected at each point of ∂D and if ∂D is not homeomorphic to $|y|=1$, then $K(D)=\infty$.*

For if $K(D)<\infty$, there would exist a quasiconformal mapping $y(x)$ of D onto $|y|<1$, Theorem 7 would imply that $y(x)$ could be extended to be a homeomorphism of $\overline{D}$ onto $|y|\leqq 1$, and hence ∂D would be homeomorphic to $|y|=1$.

There are, of course, many domains D with $K(D)<\infty$. Suppose, for example, that D is a domain which is bounded and starshaped with respect to the origin, and that

$$\log\frac{|P|}{|Q|}\leqq a\left|\frac{P}{|P|}-\frac{Q}{|Q|}\right|,\qquad 0\leqq a<\infty \tag{46}$$

for each pair of points P and Q in ∂D. Then D can be mapped quasiconformally onto $|y|<1$ by means of central projection, and it is easy to show that

$$K(D) \leqq (1+a)^{3/2}. \tag{47}$$

The inequality (46) will hold for some a if D is bounded by a finite number of compact smooth surfaces, none of which has a tangent plane passing through the origin. Hence cylinders, hemispheres, ellipsoids and convex polyhedra all have finite coefficients of quasiconformality.

In this same direction, the space form of the Carathéodory theorem on variable domains, mentioned in §14, allows us to prove the following theorem.

THEOREM 15. *If D is the union of an expanding sequence of domains $\{D_n\}$ and if D has a finite boundary point, then*

$$K(D) \leqq \liminf_{n\to\infty} K(D_n).^5 \tag{48}$$

We consider an example. Let D be the domain bounded by the parallel planes $|x_3|=1$, that is

$$D = \{x\colon (x_1^2+x_2^2)^{1/2} < \infty, \ |x_3| < 1\}. \tag{49}$$

Then D is 1-quasiconformally equivalent to a domain bounded by two spheres, one of which is internally tangent to the other, and hence $K(D)=\infty$ by Theorem 14 [27]. Next we see that D is the union of the expanding family of right cylinder domains

$$D_t = \{x\colon (x_1^2+x_2^2)^{1/2} < t, \ |x_3| < 1\}, \qquad 0<t<\infty, \tag{50}$$

and hence we conclude from Theorem 15 that

$$\lim_{t\to\infty} K(D_t) = \infty. \tag{51}$$

In contrast to the situation for the domain in (49), one can use the logarithm mapping to show that $K(D) \leqq \pi/2$ for the infinite rod

$$D = \{x\colon (x_1^2+x_2^2)^{1/2} < 1, \ |x_3| < \infty\}.$$

(See also [27] for another interesting mapping.)

One of the most important problems in the study of quasicon-

[5] Theorem 15 and the inequalities (47) and (54) are unpublished results which will appear shortly in a joint paper by J. Väisälä and the present author.

formal mappings in space is that of characterizing the domains D for which $K(D) < \infty$. This is obviously a very difficult question which probably has no simple answer. One may, however, approach this problem by trying to obtain bounds for $K(D)$ for various classes of domains D. A useful upper bound for certain elementary domains may lead to an upper bound for more complicated domains by means of Theorem 15. Similarly, a lower bound for certain classes of domains will indicate for what kinds of domains D we must expect that $K(D) = \infty$.

We conclude this paper by giving a lower bound for $K(D)$ for a class of domains D suggested by the above example. For each $t > 1$ we set

$$f(t) = \inf_{D} K(D), \tag{52}$$

where the infimum is taken over all domains D which satisfy the following conditions:

(i) D contains a disk Δ of radius t,

(ii) ∂D contains a pair of points which lie within distance 1 of the center of Δ and which are separated by the plane containing Δ.

Then one can show that the function $f(t)/\log t$ is nonincreasing in t and that

$$\lim_{t\to\infty} \frac{f(t)}{\log t} = \mu \quad \text{where} \quad .12 \leqq \mu \leqq e. \tag{53}$$

We thus conclude that

$$K(D) \geqq \mu \log t \tag{54}$$

for each domain D satisfying (i) and (ii), and that the form of this lower bound is, in a sense, best possible. In particular we see that the domains D_t in (50) satisfy (i) and (ii), and hence (51) follows directly from (54).

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