ON LEFT AMENABLE SEMIGROUPS WHICH ADMIT COUNTABLE LEFT INVARIANT MEANS

BY E. GRANIRER

Communicated by M. M. Day, July 12, 1962

Introduction. This work is closely connected with [**5**] and we shall freely use the notations of [**5**, §2].

Let G be a semigroup which admits countable left invariant means (i.e., $Ml(G) \cap Ql_1(G) \neq \emptyset$, where $Ml(G)$ is the set of left invariant means) and let $\mathfrak{g}l(G) = \{\phi;\ \phi \in m(G)^*,\ L_g\phi = \phi \text{ for } g \in G\}$. By Theorem 4.2 of [**5**] G contains finite groups which are left ideals with left cancellation, i.e. by [**5**, §2] (l.i.l.c.).[1] Let $\{A_\alpha\}_{\alpha \in I}$ be the set of all finite groups which are (l.i.l.c.) in G and define for $\alpha, \beta \in I$, $\alpha \cdot \beta = \beta$. The indices set I becomes thus a semigroup with semigroup algegra $l_1(I)$ and second conjugate algebra $m(I)^*$ (as defined in Day [**3**, p. 526]). As proved in Theorem 1 of M. M. Day [**3**, p. 530] $\mathfrak{g}l(G)$ is also a subalgebra of $m(G)^*$ (when regarded as the second conjugate algebra of $l_1(G)$).

A linear positive isometry from $m(I)^*$ *onto* $\mathfrak{g}l(G)$ which displays the inner structure of $\mathfrak{g}l(G)$ is constructed in this paper. This isometry is also an algebraic isomorphism from the algebra $m(I)^*$ onto the algebra $\mathfrak{g}l(G)$.

$A = \bigcup_{\alpha \in I} A_\alpha$ is a right minimal ideal (this is the result of Clifford [**1**,] for proof see [**5**, Lemma 3.1 and Remark 3.1]) and moreover, the A_α's as finite groups are isomorphic to one another (see [**6**] end of proof of Theorem E) therefore the number N of elements of A_α does not depend on α. We now define the linear operator $T: m(G) \to m(I)$:

$$\text{for } \alpha \in I \qquad (Tf)(\alpha) = \frac{1}{N} \sum_{g \in A_\alpha} f(g).$$

This operator has the following properties:

(1) If $f(g) \geq 0$ for each g in G then $(Tf)(\alpha) \geq 0$ for each $\alpha \in I$ (obvious).

(2) $T1_G = 1_I$ (obvious).

(3) $T(l_a f) = T(f)$ for each a in G and f in $m(G)$:

$$(Tl_a f)(\alpha) = \frac{1}{N} \sum_{g \in A_\alpha} (l_a f)(g) = \frac{1}{N} \sum_{g \in A_\alpha} f(ag) = \frac{1}{N} \sum_{g \in A_\alpha} f(g) = (Tf)(\alpha)$$

($aA_\alpha = A_\alpha$ since A_α is a finite (l.i.l.c.))

[1] A finite group $B \subset G$ is a (l.i.l.c.) if $gB = B$ for each g in G.

(4) T is linear and $\|Tf\| \leqq \|f\|$ for f in $m(G)$. Linearity is evident and

$$\|Tf\| = \sup_{\alpha\in I} |(Tf)(\alpha)| = \sup_{\alpha\in I} \frac{1}{N}\left|\sum_{g\in A_\alpha} f(g)\right| \leqq \sup_{\alpha\in I} \frac{1}{N} \sum_{g\in A_\alpha} |f(g)|$$
$$\leqq \frac{1}{N} \sum_{g\in A_\alpha} \|f\| = \|f\|.$$

(5) $T[m(G)] = m(I)$ since if h is in $m(I)$ then we define f in $m(G)$ as follows: for $g \in A_\alpha$ let $f(g) = h(\alpha)$ and this for each $\alpha \in I$, for g not in $A = \bigcup_{\alpha\in I} A_\alpha$ (if there exist at all such g's) let $f(g) = 0$. Obviously f is in $m(G)$ and:

$$(Tf)(\alpha) = \frac{1}{N} \sum_{g\in A_\alpha} f(g) = \frac{1}{N} \sum_{g\in A_\alpha} h(\alpha) = h(\alpha).$$

Thus $Tf = h$, but moreover the above chosen f satisfies also $\|f\| \leqq \|h\|$. Thus the image of the closed unit ball in $m(G)$ by T, is the whole closed unit ball of $m(I)$.

(6) If $B \subset G$ is a finite set then $(T1_B)(\alpha)$ does not vanish at most on a finite subset of I. In fact $(T1_B)(\alpha) = (1/N)\sum_{g\in A_\alpha} 1_B(g)$, thus $T(1_B)$ does not vanish only for those α which satisfy $B \cap A_\alpha \neq \emptyset$. Since B is finite there is at most a finite number of such α.

If S is a set then let $c_0(S)^\perp \subset m(S)^*$ be defined by

$$c_0(S)^\perp = \{\phi;\ \phi(1_g) = 0 \text{ for each } g \text{ in } S\}$$

We are now ready to prove the following:

THEOREM. *$T^*: m(I)^* \to m(G)^*$ is a linear positive isometry from $m(I)^*$ onto $\mathcal{gl}(G)$ such that $T^*[Ql_1(I)] = Ql_1(G) \cap \mathcal{gl}(G)$ and*

$$T^*[c_0(I)^\perp] = c_0(G)^\perp \cap \mathcal{gl}(G).$$

PROOF. $(T^*\phi)f = \phi(Tf)$ for ϕ in $m(I)^*$ and f in $m(G)$. T^* is linear and moreover is isometric since:

$$\|T^*\phi\| = \sup_{f\in m(G), \|f\|\leqq 1} |(T^*\phi)(f)| = \sup_{f\in m(G), \|f\|\leqq 1} |\phi(Tf)|$$
$$=^{(*)} \sup_{h\in m(I), \|h\|\leqq 1} |\phi(h)| = \|\phi\|$$

(for (*) see (5) above). Now for ϕ in $m(I)^*$, f in $m(G)$ and a in G

$$(T^*\phi)(l_a f) = \phi(Tl_a f) = \phi(Tf) = (T^*\phi)(f)$$

(see (3) above) which implies that $T^*(m(I)^*) \subset \mathcal{gl}(G)$. We prove now that $T^*(m(I)^*) = \mathcal{gl}(G)$. In order to do this we have to prove at first that if ϕ is in $\mathcal{gl}(G)$ and f in $m(G)$ is such that $Tf=0$ (i.e., $Tf(\alpha) = (1/N)\sum_{g\in A_\alpha} f(g)=0$ for each $\alpha\in I$) then $\phi(f)=0$. In fact if $G-A$ will denote those elements of G which are not in $A=\bigcup_{\alpha\in I} A_\alpha$ (this may be an empty set) and if a is some element of A, then for any f' of $m(G)$ and g of G we have

$$(l_a(f'1_{G-A}))(g) = (f'1_{G-A})(ag) = f'(ag)1_{G-A}(ag) = 0$$

since ag belongs to A for any g of G (see remark (3.1) of [5]). Thus:

$$\phi(f') = \phi(f'1_A) + \phi(f'1_{G-A}) = \phi(f'1_A) + \phi(l_a(f'1_{G-A})) = \phi(f'1_A).$$

Let us pick some α_0 of I and let $a_1, \cdots, a_N$ be the N elements of A_{α_0}. If a is an arbitrary but fixed element of A then $a\in A_\alpha$ for some $\alpha\in I$. But $A_{\alpha_0}\cdot a$ is a left ideal and $A_{\alpha_0}\cdot a\subset A_\alpha$. Since A_α is a minimal left ideal (as a left ideal and group) $A_{\alpha_0}\cdot a=A_\alpha$. Now

$$\left[\frac{1}{N}\sum_{i=1}^{N} l_{a_i} f\right](a) = \frac{1}{N}\sum_{i=1}^{N} f(a_i a) = \frac{1}{N}\sum_{g\in A_\alpha} f(g) = (Tf)(\alpha).$$

But by assumption $(Tf)(\alpha)=0$ for each $\alpha\in I$, which implies that

$$\left[\frac{1}{N}\sum_{i=1}^{N} l_{a_i} f\right](g) = 0 \qquad \text{for each } g \text{ of } A.$$

But since ϕ is left invariant

$$\phi(f) = \phi\left[\frac{1}{N}\sum_{i=1}^{N} l_{a_i} f\right] = \phi\left[\left(\frac{1}{N}\sum_{i=1}^{N} l_{a_i} f\right)1_A\right] = \phi(0) = 0$$

which proves that if $Tf=0$ for some f of $m(G)$ then $\phi(f)=0$ for each ϕ of $\mathcal{gl}(G)$.

Let now ϕ_0 be an arbitrary but fixed element of $\mathcal{gl}(G)$. We define ψ_0 of $m(I)^*$ such that $T^*\psi_0=\phi_0$ as follows: if h is in $m(I)$ then let $f\in m(G)$ be such that $Tf=h$ (by (5) above there exists such an f). We define $\psi_0(h)=\phi_0(f)$. ψ_0 is well defined on $m(I)$, since if f_1 is such that $Tf_1=h=Tf$ then $T(f_1-f)=0$ and thus $\phi_0(f_1-f)=0$. We get that $\phi_0(f_1)=\phi_0(f)=\psi_0(h)$.

ψ_0 is linear since if $h_i=Tf_i$, $i=1, 2$, then $T(\alpha f_1+\beta f_2)=\alpha Tf_1+\beta Tf_2$ and $\psi_0(\alpha h_1+\beta h_2)=\phi_0(\alpha f_1+\beta f_2)=\alpha\phi_0(f_1)+\beta\phi_0(f_2)=\alpha\psi_0(h_1)+\beta\psi_0(h_2)$.

If h is in $m(I)$ then we can choose by (5) above a f in $m(G)$ such that $\|f\|\leqq\|h\|$ and $Tf=h$. Thus $|\psi_0(h)|=|\phi_0(f)|\leqq\|\phi_0\|\,\|f\|\leqq\|\phi_0\|\,\|h\|$ which implies that ψ_0 is in $m(I)^*$. But for f in $m(G)$ let $h=Tf$, then $(T^*\psi_0)(f)=\psi_0(Tf)=\psi_0(h)=\phi_0(f)$ which proves that $T^*\psi_0=\phi_0$, in other words that T^* is a linear isometry from $m(I)^*$ *onto* $\mathcal{gl}(G)$.

If now $\psi\in m(I)^*$ is non-negative (i.e., $\psi(h)\geqq 0$ for $h\geqq 0$) and if

$f \in m(G)$ is non-negative then by (1) above Tf is non-negative and $(T^*\psi)(f) = \psi(Tf) \geqq 0$ which proves that T^* is positive.

If $\psi \in c_0(I)^\perp$ (i.e., $\psi(1_\alpha) = 0$ for each $\alpha \in I$) and if $B \subset G$ is a finite set then $(T^*\psi)(1_B) = \psi(T1_B) = 0$, since by (6) above $T(1_B)$ does not vanish at most on a finite set. Thus $T^*\psi$ is in $c_0(G)^\perp$ and $T^*(c_0(I)^\perp) \subset c_0(G)^\perp \cap \mathfrak{gl}(G)$.

If now, ψ is in $Ql_1(I)$ ($\|\psi\| = \sum_{\alpha \in I} |\psi(1_\alpha)| < \infty$) then we define ϕ in $Ql_1(G)$ as follows:

$\phi(1_g) = \psi(1_\alpha)$ for $g \in A_\alpha$, $\alpha \in I$ and $\phi(1_g) = 0$ for $g \in G - A$ (if non-void). $\psi(1_\alpha) \neq 0$ at most on a countable subset of I, therefore $\phi(1_g) \neq 0$ at most on a countable subset of G and

$$\sum_{g \in G} |\phi(1_g)| = \sum_{\alpha \in I} \sum_{g \in A_\alpha} |\phi(1_g)| = \sum_{\alpha \in I} N |\psi(1_\alpha)| = N\|\psi\| < \infty.$$

Moreover,

$$(T^*\psi)(f) = \psi(Tf) = \sum_{\alpha \in I} \psi(1_\alpha)(Tf)(\alpha) = \sum_{\alpha \in I} \psi(1_\alpha) \frac{1}{N} \sum_{g \in A_\alpha} f(g)$$
$$= \sum_{g \in G} \frac{1}{N} \phi(1_g) f(g)$$

which implies that $T^*\psi = (1/N)\phi$ and $\phi \in Ql_1(G)$. Thus $T^*(Ql_1(I)) \subset Ql_1(G) \cap \mathfrak{gl}(G)$.

If $\phi \in c_0(G)^\perp \cap \mathfrak{gl}(G)$ then there is some $\psi \in m(I)^*$ such that $T^*\psi = \phi$. However as well known, $c_0(I)^\perp \oplus Ql_1(I) = m(I)^*$ (see [7, p. 429]), which implies that ψ can be decomposed into $\psi = \psi_1 + \psi_2$ with $\psi_1 \in Ql_1(I)$ and $\psi_2 \in c_0(I)^\perp$. Thus $T^*\psi = T^*\psi_1 + T^*\psi_2 = \phi$. But from above, $T^*\psi_2 \in c_0(G)^\perp$ and by assumption $\phi \in c_0(G)^\perp$, which implies that $T^*\psi_1 \in c_0(G)^\perp \cap Ql_1(G) = \{0\}$.

We have shown that $T^*(c_0(I)^\perp) = c_0(G)^\perp \cap \mathfrak{gl}(G)$. In the same way one gets that $T^*(Ql_1(I)) = Ql_1(G) \cap \mathfrak{gl}(G)$ which finishes the proof of the theorem.

Remarks. If $\phi_1, \phi_2 \in m(I)^*$ and $x \in m(I)$ then $(\phi_1 \odot \phi_2)(x) = \phi_1(\phi_2 l'_\alpha x)$ where l'_α is the left translation operator in $m(I)$ with respect to the element $\alpha \in I$. (See M. M. Day [3, p. 527].) Since $(l'_\alpha x)(\beta) = x(\alpha\beta) = x(\beta)$ one gets that $(\phi_1 \odot \phi_2)(x) = \phi_1(\phi_2(x)1_I) = \phi_1(1_I) \cdot \phi_2(x)$ and thus $\phi_1 \odot \phi_2 = \phi_1(1_I)\phi_2$, which implies that $T^*(\phi_1 \odot \phi_2) = \phi_1(1_I) T^*\phi_2$. (Until now $\odot$ denoted multiplication in $m(I)^*$. From now on its denotes multiplication in $m(G)^*$.) But $(T^*\phi_1) \odot (T^*\phi_2) = ((T^*\phi_1)(1_G)) T^*\phi_2$ (see Day [3, p. 530]). Since $(T^*\phi_1)(1_G) = \phi_1(T1_G) = \phi_1(1_I)$ one gets that $T^*(\phi_1 \odot \phi_2) = (T^*\phi_1) \odot (T^*\phi_2)$ which implies the

Corollary. *T^* is an algebraic isomorphism from $m(I)^*$ onto $\mathfrak{gl}(G)$.*

REMARKS. If $\phi \in \mathfrak{gl}(G)$ then let $\psi=(T^*)^{-1}(\phi)$. By Jordan's decomposition theorem $\psi=\psi_1-\psi_2$ with non-negative ψ_1, ψ_2 of $m(I)^*$ (see [4, p. 98]). Thus $\phi=T^*\psi=T^*\psi_1-T^*\psi_2$ where $T^*\psi_i$ $i=1, 2$ are non-negative disjoint left invariant elements in $m(G)^*$. If $\alpha_i=(T^*\psi_i)(1_G) \geqq 0$ then it is easily seen that ϕ can be written as $\phi=\alpha_1\phi_1-\alpha_2\phi_2$ where ϕ_1, ϕ_2 are either left invariant means or zero. ($\phi_i=(1/\alpha_i)T^*\psi_i$ if $\alpha_i>0$ and $\phi_i=0$ if $\alpha_i=0$.) For semigroups with cancellation, this is a result of M. M. Day [2, p. 281].

REFERENCES

1. A. H. Clifford, *Semigroups containing minimal left ideals*, Amer. J. Math. **70** (1948), 521–526.

2. M. M. Day, *Means for the bounded functions and ergodicity of the bounded representations of semigroups*, Trans. Amer. Math. Soc. **69** (1950), 276–291.

3. ———, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.

4. N. Dunford and J. Schwartz, *Linear operators*. I, Interscience, N. Y., 1958.

5. E. Granirer, *On amenable semigroups with a finite dimensional set of invariant means*. I, Illinois J. Math. (to appear).

6. ———, *On amenable semigroups with a finite dimensional set of invariant means*. II, Illinois J. Math. (to appear).

7. G. Köthe, *Topologische lineare Räume*, Springer, Berlin, 1960.

THE HEBREW UNIVERSITY OF JERUSALEM