OPERATOR AVERAGES[1]

BY R. V. CHACON

Communicated by G. A. Hedlund, February 7, 1962

The object of this note is to state a general result the proof of which appears in [4] and which includes the ergodic theorem proved in [1] as well as the Hopf-Dunford-Schwartz ergodic theorem. Let $(S, \mathfrak{F}, \mu)$ be a σ-finite measure space, that is, let S be a set of points s, $\mathfrak{F}$ a Borel field of subsets A of S and μ a σ-finite measure defined on $\mathfrak{F}$. Let L_1 be the Banach space of complex-valued integrable functions $f(s)$ having S for their domain of definition. Dunford and Schwartz [5] have extended Hopf's ergodic theorem [6] as follows:

THEOREM. *Let T be a linear operator of L_1 to L_1 with $\|T\| \leqq 1$, and with* $(\|T\|_\infty \leqq 1)$

$$\operatorname{ess.\,sup.}_{s\in S} | Tg(s) | \leqq \operatorname{ess.\,sup.}_{s\in S} | g(s) |, \text{ each } g \in L_1 \cap L_\infty.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(s)$$

exists almost everywhere, for f in L_1.

Ornstein and the author [1] proved the following theorem which was conjectured by Hopf [6]:

THEOREM. *Let T be a linear operator of L_1 to L_1 with $\|T\| \leqq 1$ and with $T \geqq 0$. Then*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} T^k f(s)}{\sum_{k=0}^{n} T^k p(s)}$$

exists almost everywhere on $\{s: 0 < \sum_{k=0}^{\infty} T^k p(s) \leqq +\infty\}$ for f and p in L_1, $p(s) \geqq 0$ almost everywhere.

It is of course clear that neither result contains the other and that

[1] The work reported in this paper was carried out under a grant from the National Science Foundation.

both contain the classical ergodic theorems. In the present note we state an ergodic theorem which contains the above theorems, and which allows, for example, the consideration of ratios of the form

$$\frac{\sum_{k=0}^{n} e^{ik\alpha}T^k f(s)}{\sum_{k=0}^{n} T^k p(s)}$$

where T is a linear transformation of L_1 to L_1 with $\|T\| \leqq 1$ and with $T \geqq 0$. Our result is the following:

THEOREM. *Let $\{p_n\}$ be a sequence of non-negative measurable functions and let T be a linear operator of L_1 to L_1 with $\|T\| \leqq 1$ and with $|Tg(s)| \leqq p_{n+1}(s)$ almost everywhere whenever $|g(s)| \leqq p_n(s)$ almost everywhere and $g \in L_1$. Then*

$$\lim_{n \to \infty} \frac{\sum_{k=0}^{n} T^k f(s)}{\sum_{k=0}^{n} p_k(s)}$$

exists almost everywhere on $\{s: 0 < \sum_{k=0}^{\infty} p_k(s) \leqq +\infty\}$.

To see that the Hopf-Dunford-Schwartz theorem is implied by this result, note that the condition $\|T\|_\infty \leqq 1$ may be written $|Tg(s)| \leqq 1$ almost everywhere whenever $|g(s)| \leqq 1$ almost everywhere and $g \in L_1$, and thus we have the theorem on taking $p_n(s) \equiv 1$, $s \in S$, $n = 1, 2, 3, \cdots$.

If $T \geqq 0$ then $|Tg(s)| \leqq T^{n+1}p(s)$ almost everywhere whenever $|g(s)| \leqq T^n g(s)$ almost everywhere and $g \in L_1$ and thus we have the second theorem on taking $p_n(s) = T^n p(s)$, $n = 1, 2, 3, \cdots$.

The theorem is proved by showing that to each function f in L_1 there corresponds a function f^* in L_1 and a function $\alpha(s)$ with $|\alpha(s)| = 1$ such that f^* has the same limiting behavior as f and such that

$$\frac{T^n g(s)}{|T^n g(s)|} = \alpha(s) \text{ on } \{s: |T^n g(s)| > 0\}$$

for any function g in L_1 having support contained in the support of f^* and having the additional property that

$$\frac{g(s)}{|g(s)|} = \frac{f^*(\alpha)}{|f^*(s)|}.$$

The existence of such functions is obtained by a kind of uniform convexity argument together with a lemma which unites the maximal ergodic lemmas of [1] and [2] as well as sharpening them. After the operator T is reduced in this way the proof is completed utilizing the results of [3] and of [1].

Bibliography

1. R. V. Chacon and D. S. Ornstein, *A general ergodic theorem*, Illinois J. Math. **4** (1960), 153–160.

2. R. V. Chacon, *On the ergodic theorem without assumption of positivity*, Bull. Amer. Math. Soc. **67** (1961), 181–190.

3. ———, *Identification of the limit of operator averages*, J. Math. Mech (to appear).

4. ———, *Convergence of operator averages*, to appear.

5. N. Dunford and J. T. Schwartz, *Convergence almost everywhere of operator averages*, J. Math. Mech. **5** (1956), 129–178.

6. E. Hopf, *The general temporally discrete Markoff process*, J. Math. Mech. **3** (1954), 13–45.

Brown University