ON THE EXTREME EIGENVALUES OF TRUNCATED TOEPLITZ MATRICES

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Let $f(\theta)$ be a real-valued Lesbesgue integrable function defined on $[-\pi, \pi]$. Let $\{C_j\}$ be the Fourier coefficients of $f(\theta)$, i.e.,

$$f(\theta) \sim \sum_{-\infty}^{\infty} C_j e^{ij\theta}.$$

The matrix $T_n[f] = (C_{s-j})$; $s, j = 0, 1, \cdots, n$ is the nth finite section of the infinite Toeplitz matrix (C_{s-j}) associated with the function $f(\theta)$.

In this note we are concerned with functions $f(\theta)$ satisfying

CONDITION A. Let $f(\theta)$ be real, continuous and periodic with period 2π. Let $\min f(\theta) = f(0) = m$ and let $\theta = 0$ be the only value of $\theta(\text{mod } 2\pi)$ for which this minimum is attained.

CONDITION A(α). Let $f(\theta)$ be a function satisfying condition A. Moreover, let $f(\theta)$ have continuous derivatives of order 2α in some neighborhood of $\theta = 0$. Finally let $f^{(2\alpha)}(0) = \sigma^2 > 0$ be the first non-vanishing derivative of $f(\theta)$ at $\theta = 0$.

THEOREM. *Let $f(\theta)$ satisfy conditions A and $A(\alpha)$. Let $\lambda_{\nu,n}$ ($\nu = 1, 2, \cdots, n+1$) be the eigenvalues of $T_n[f]$ arranged in non-decreasing order. For fixed ν, as $n \to \infty$ we have*

$$\lambda_{\nu,n} = m + \frac{\sigma^2}{(2\alpha)!} \Lambda_\nu \left(\frac{1}{n}\right)^{2\alpha} + o\left(\frac{1}{n}\right)^{2\alpha}, \tag{1}$$

where the numbers Λ_ν are the eigenvalues arranged in nondecreasing order of

$$\left[-\left(\frac{d}{dx}\right)^2\right]^\alpha U - \Lambda U = 0, \qquad 0 \leqq x \leqq 1, \tag{2}$$

with boundary conditions

$$\left(\frac{d}{dx}\right)^i U(0) = \left(\frac{d}{dx}\right)^i U(1) = 0, \qquad i = 0, 1, \cdots, \alpha - 1. \tag{2a}$$

The case $\alpha = 1$ was studied by Kac, Murdock and Szegö [3]. In [5] Widom also studied the case $\alpha = 1$ and, under suitable conditions, obtained the next term in the asymptotic expansion of $\lambda_{\nu,n}$. The case $\alpha = 2$ was studied by this author [4].

The validity of this theorem was conjectured by Widom [6]. In fact, his conjecture is much more general.

The author is indebted to Professors Kac and Widom for many fruitful discussions concerning these problems.

In view of the Weyl-Courant characterization of $\lambda_{\nu,n}$ (and Λ_ν) as solutions of a variational problem, it is sufficient to consider the case where $f(\theta)$ is an even trigonometric polynomial. (See [4] or [5] for a more detailed argument.) Moreover, there is no loss in generality in assuming $m=0$. Thus $f(\theta)$ may be written as

$$f(\theta) = \beta_0(1 - \cos\theta)^\alpha + \sum_{k=1}^{N} \beta_k(1 - \cos\theta)^{k+\alpha} \tag{3}$$

where

$$\beta_0 = \frac{2^\alpha \sigma^2}{(2\alpha)!} \cdot \tag{3a}$$

Let us interpret the eigenvalue problem as a difference equation. Let $R=N+\alpha-1$ and let D_n be the interval $[-R/(n+2),\ 1+R/(n+2)]$. Let $\Delta x=1/(n+2)$ and let $x_j=j\Delta x$ be the lattice points in D_n. If $\phi(x)$ is any function defined on D_n we denote $\phi(x_j)$ by ϕ_j.

Let P_n be the class of piecewise-linear functions $h(x)$ defined on D_n and determined by their values at x_j which satisfy

$$h_j = 0 \quad \text{for} \quad j \leqq 0 \quad \text{and} \quad \geqq n + 2. \tag{4}$$

Let

$$T_n[(1 - \cos\theta)^r] = \tau_r \tag{4.1}$$

and let δ be the second central divided difference operator, i.e.,

$$(\delta\phi)_j = \left(\frac{1}{\Delta x}\right)^2 \{\phi_{j+1} - 2\phi_j + \phi_{j-1}\}. \tag{4.2}$$

We observe that every function $h(x) \in P_n$ corresponds to an $(n+1)$ vector $H=(h_j)$, $j=1, 2, \cdots, n+1$, and conversely.

Furthermore, it is easy to relate the matrices $\tau_r (r \leqq R+1)$ to the operator δ. We have

$$(\tau_r H)_j = \left(-\frac{1}{2}\Delta x^2\right)^r (\delta^r h)_j, \qquad = 1, 2, \cdots n + 1, \tag{4.3}$$

thus

$$(4.4)\qquad (T_n[f]H)_j = \beta_0\left(-\frac{1}{2}\Delta x^2\right)^{\alpha}(\delta^{\alpha}h)_j + \sum_{k=1}^{N}\left(-\frac{1}{2}\Delta x^2\right)^{\alpha+k}\beta_k(\delta^{\alpha+k}h)_j.$$

Let S_n be the finite difference operator which corresponds to $(n+2)^{2\alpha}T_n[f]$, i.e.,

$$(4.5)\qquad S_n = \left(-\frac{1}{2}\right)^{\alpha}\beta_0\delta^{\alpha} + \sum_{k=0}^{N}\left(-\frac{1}{2}\Delta x^2\right)^{k}\beta_k\delta^{\alpha+k}.$$

Clearly, S_n is a consistent approximation to the differential operator

$$(4.5\text{a})\qquad \frac{\sigma^2}{(2\alpha)!}\left[-\left(\frac{d}{dx}\right)^2\right]^{\alpha}.$$

Thus our theorem is seen to be equivalent to the theorem that the eigenvalues $\Lambda_{\nu,n}$ of S_n acting on functions $h(x)\in P_n$ converge to the eigenvalues of (4.5a) subject to the boundary conditions (2a).

We require one more definition. Let $h(x)$, $g(x)\in P_n$, let H and G be the corresponding $(n+1)$ vectors, then

$$(4.6)\qquad [h, g] \equiv \Delta x \sum h_j g_j = \Delta x(H, G).$$

Lemma 1.

$$(5.1)\qquad \operatorname*{Lim\,Sup}_{n\to\infty} \Lambda_{\nu,n} = \operatorname*{Lim\,Sup}_{n\to\infty} (n+2)^{2\alpha}\lambda_{\nu,n} \leqq \frac{\sigma^2}{(2\alpha)!}\Lambda_{\nu}.$$

Proof. This follows immediately from the Weyl-Courant characterization of $\lambda_{\nu,n}$ and the appropriate choice of "test" vectors obtained from the eigenfunctions of (2). (See Weinberger [7] where this is carried out in detail for a similar problem.)

Let $\Delta(\alpha)$ be the divided-difference operator of order α determined as follows:

$$(\text{a})\qquad \alpha = 2\gamma\colon\ \Delta(\alpha) = \delta^{\gamma}$$

and

$$(\text{b})\qquad \alpha = 2\gamma + 1\colon\ \Delta(\alpha) = \delta^{\gamma}\cdot D$$

where D is a first order divided-difference operator (forward or backward, it doesn't matter).

Lemma 2. *Let H be an eigenvector of $T_n[f]$ associated with $\lambda_{\nu,n}$ and let $h(x)\in P_n$ be the associated function with $h(x)$ (i.e., H) normalized so that $[h, h]=1$.*

There exists a constant M_ν independent of n, such that

$$[\Delta(\alpha)h,\ \Delta(\alpha)h] \leqq M_\nu. \tag{5.2}$$

PROOF. We first prove that

$$[(-\delta)^\alpha h,\ h] \leqq M_\nu, \tag{5.2a}$$

and (5.2) follows from α applications of summation by parts. (Note: $-\delta$ is a positive definite operator.)

However, (5.2a) is equivalent to

$$\Delta x\cdot 2^\alpha (n+2)^{2\alpha}(\tau_\alpha H,\ H) \leqq M_\nu. \tag{5.2b}$$

Now, Lemma 1 implies the existence of a constant L_ν such that

$$\Delta x(n+2)^{2\alpha}(T_n[f]H,\ H) \leqq L_\nu. \tag{5.3}$$

However, as is well known (see [4] or [5]), if $\phi(\theta)=\sum_{j=1}^{n+1} h_j e^{i(j-1)\theta}$, then

$$(T_n[f]H,\ H) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,|\,\phi\,|^2 d\theta, \tag{5.3a}$$

and

$$(\tau_\alpha H,\ H) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (1-\cos\theta)^\alpha\,|\,\phi\,|^2 d\theta. \tag{5.3b}$$

We write $f(\theta)$ as $f(\theta)=(1-\cos\theta)^\alpha Q(\theta)$, where

$$Q(\theta) = \beta_0 + \sum_{k=1}^{N} \beta_k (1-\cos\theta)^k.$$

Since $f(\theta)$ satisfies conditions A and $A(\alpha)$, there is a positive constant Q_0 such that

$$0 < Q_0 \leqq Q(\theta).$$

Thus, (5.3a), (5.3b) together with (5.3) implies

$$2^\alpha \Delta x(n+2)^{2\alpha}(\tau_\alpha H,\ H) \leqq 2^\alpha\cdot L_\nu/Q_0,$$

which proves the lemma.

Using Lemma 2 and more-or-less standard techniques in Analysis (see Courant, Friedrichs and Lewy [1]) one readily obtains the following result on the compactness of the eigenfunctions $h(x)\in P_n$.

LEMMA 3. *Let $\{H_{\nu,n}\}$ be a sequence of eigenvectors of $T_n[f]$ associated with $\lambda_{\nu,n}$. Let $H\equiv\{h_n(x)\}$ be the associated sequence of functions in P_n. There exists a subsequence $\{h_{n'}(x)\}$ which converges uniformly on $[0, 1]$ to a function $u(x)$. In addition, $u(x)$ has $(\alpha-1)$ continuous derivatives* *and has strong derivatives of order α which satisfy*

$$\int_0^1 |u^{(\alpha)}|^2 dx \leqq M_\nu.$$

Moreover, the divided-difference of $h_{n'}(x)$ of order k with $k \leqq \alpha - 1$ also converge uniformly to the kth derivative of $u(x)$. Finally, in virtue of this last statement

$$u^{(k)}(0) = u^{(k)}(1) = 0, \qquad k = 0, 1, 2, \cdots, \alpha - 1.$$

Our proof is almost complete. Let $\phi(x)$ be any function in $C_\infty[0, 1]$ which satisfies the boundary conditions (2a). We may extend ϕ as a C_∞ function in D_n. There is no confusion if we also call this extended function ϕ. Also, given such a function $\phi(x)$ we may construct a function $\hat{\phi} \in P_n$ in the obvious way.

Consider the sequence $H = \{h_n(x)\}$ associated with $\lambda_{\nu,n}$. We may choose a subsequence $\{h_{n'}(x)\}$ so that $\Lambda_{\nu,n'} = (n'+2)^{2\alpha}\lambda_{\nu,n'}$ converge to a value Λ_ν^0. We may now choose a subsequence (in accordance with Lemma 3) so that the $h_{n''}(x) \to u(x)$. We write n for n'', and proceed.

LEMMA 4. *Let $\phi \in C_\infty[0, 1]$, then*

$$[S_n h_n, \hat{\phi}] = \frac{\sigma^2}{\alpha!}(-1)^\alpha \int_0^1 u(x)\left(\frac{d}{dx}\right)^{2\alpha} \phi \cdot dx + o(1).$$

PROOF. Let Φ be the $(n+1)$ vector associated with $\hat{\phi}$, then, since $T_n[f]$ is hermitian,

$$\begin{aligned}[S_n h_n, \phi] &= \Delta x(n + 2)^{2\alpha}(T_n[f]H_n, \Phi) \\ &= \Delta x(n + 2)^{2\alpha}(H_n, T_n[f]\Phi).\end{aligned} \tag{5.4}$$

For any point x_j for which $R+1 < j < (n+2) - (R+1)$, Taylor's theorem gives us

$$(n + 2)^{2\alpha}(T_n[f]\Phi)_j = \frac{\sigma^2}{\alpha!}(-1)^\alpha\left(\frac{d}{dx}\right)^{2\alpha}\phi + O(\Delta x^2) \tag{5.5a}$$

Consider now any other point x_j, $1 \leqq j \leqq n+1$. Let $\alpha \leqq r \leqq R+1$, then

$$\begin{aligned}(n + 2)^{2\alpha}(\tau_r\Phi)_j = \left(-\frac{1}{2}\right)^r (\Delta x)^{2(r-\alpha)}\left[\left(\frac{d}{dx}\right)^{2r}\phi\right]_j \\ + O\left[\phi_j\left(\frac{1}{\Delta x}\right)^{2\alpha}\right].\end{aligned} \tag{5.5b}$$

Since $\phi_j = O(\Delta x^\alpha)$, the error term in (5.5b) is $O[(1/\Delta x)^\alpha]$. Since $h_j = o(\Delta x^{\alpha-1})$ we find the error in the contribution to 5.4, i.e., the error in

$$\Delta x(n+2)^{2\alpha} h_j(\sigma_r \Phi)_j,$$

is $o(1)$. Thus our lemma is proven.

However, we also have

$$[S_n h_n, \phi] \to \Lambda_\nu^0 \int_0^1 u(x)\phi(x)dx \tag{5.6}$$

which, together with Lemma 4 implies that $u(x)$ is a "weak" eigenfunction (with eigenvalue Λ_ν^0) of the operator (4.5a). But, upon considering the equivalent integral equation (using the Green's function), we see that such a weak eigenfunction is indeed an eigenfunction with eigenvalue Λ_ν^0.

However, Lemma 1 and the Weyl-Courant lemma, and the uniqueness of the eigenvalues of (4.5a) show

$$\Lambda_\nu^0 = \frac{\sigma^2}{(2\alpha)!}\Lambda_\nu.$$

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