SOME THEOREMS CONCERNING FUNCTION ALGEBRAS

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In what follows, C will be a compact Hausdorff space, and $\mathfrak{A}$ will be a uniformly closed separating algebra of continuous complex-valued functions on C. That is, sums, products, and complex multiples of elements of $\mathfrak{A}$ are in $\mathfrak{A}$, uniform limits of elements of $\mathfrak{A}$ are in $\mathfrak{A}$, and for distinct points x and y in C there exists f in $\mathfrak{A}$ with $f(x) \neq f(y)$. Šilov (see [1]) has shown that there exists a smallest closed subset B of C, called the Šilov boundary of $\mathfrak{A}$, such that for each f in $\mathfrak{A}$ there exists x in B with $|f(x)| = \|f\|$, where $\|f\| = \max \{|f(y)| : y \in C\}$. The following theorem generalizes this result, in case C is metric.

THEOREM 1. *Let C be a compact metric space, and let $\mathfrak{A}$ be a uniformly closed separating algebra of continuous complex-valued functions on C. Then there exists a smallest subset M of C, called the minimal boundary of $\mathfrak{A}$, such that for each f in $\mathfrak{A}$ there exists x in M with $|f(x)| = \|f\|$. The set M is equal to the set M_0 which is defined as follows:*

$$M_0 = \{x : x \in C,\ \exists f \text{ in } \mathfrak{A} \text{ with } |f(x)| > |f(y)| \text{ for all } y \neq x \text{ in } C\}.$$

The closure of M is the Šilov boundary of $\mathfrak{A}$.

The question of the topological structure of M is answered by the following theorem.

THEOREM 2. *Let ρ be the metric on C. For each positive integer n, let*

$$U_n = \{x : x \in C,\ \exists f \text{ in } \mathfrak{A} \text{ with } \|f\| \leqq 1,\ |f(x)| > 3/4, \text{ and } |f(y)| < 1/4 \text{ for } \rho(x, y) \geqq n^{-1}\}.$$

Then $\bigcap U_n = M$.

Since it is easy to see that U_n is open, it follows that M is a G_δ.

It is known that every bounded linear functional ϕ on $\mathfrak{A}$ can be represented by a complex-valued Borel measure μ with $\|\mu\| = \|\phi\|$ on the Šilov boundary B of $\mathfrak{A}$. It is conjectured that μ can be taken to be a measure on the minimal bounary of $\mathfrak{A}$. The following theorem constitutes an important case of this conjecture.

THEOREM 3. *Let x be a point of C which is not in the minimal boundary M of $\mathfrak{A}$. Let $\mathfrak{A}$ contain the constant functions. Then there exists a non-negative Borel measure μ on $B - \{x\}$ of norm 1 such that $f(x) = \int f d\mu$ for all f in $\mathfrak{A}$.*

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As an application of these concepts, we have the following theorem.

THEOREM 4. *Let C be a compact subset of the complex plane without interior. Let $\mathfrak{A}$ be the algebra of all continuous functions on C which are uniform limits of rational functions with poles in $-C$. Let M be the minimal boundary of $\mathfrak{A}$. Let $\mathfrak{B}$ be the set of all continuous real functions on C which are uniform limits of the real parts of rational functions with poles in $-C$. Let N be the set*

$$\{x: x \in C, \exists f \text{ in } \mathfrak{B} \text{ with } |f(x)| > |f(y)| \text{ for all } y \neq x \text{ in } C\}.$$

Then $M = N$ and the following conditions are equivalent:

(i) *$\mathfrak{A}$ consists of all continuous functions on C,*

(ii) *$\mathfrak{B}$ consists of all continuous real functions on C,*

(iii) *$M = C$,*

(iiii) *$C - M$ has measure* 0.

Theorem 4 gives a necessary and sufficient condition on a compact subset C of the complex plane that every continuous function on C be uniformly approximable by rational functions. Mergelyan [2] has given sufficient conditions.

Added in proof. The conjecture preceding Theorem 3 has been proved.

REFERENCES

1. L. H. Loomis, *Abstract harmonic analysis*, New York, 1953.

2. S. N. Mergelyan, *On the representation of functions by series of polynomials on closed sets*, Amer. Math. Soc. Translations, no. 85, 1953.

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