BOOK REVIEWS

Eine neue Methode in der Analysis und deren Anwendungen. By Paul Turán. Budapest, Akadémiai Kiadó, 1953. 195+1 pp. and errata. 80.00 Ft.

This is a connected account, with extensions, of a method developed by the author in a series of papers beginning in 1941. The general theme is the application of Diophantine approximation to analysis and analytical theory of numbers. In the extensive literature that has grown up round this subject during the present century, arithmetic and analysis have become linked in a mutual relationship. Arithmetical theorems on the simultaneous approximation to real numbers by integers have been applied to the proof of inequalities for trigonometrical sums

$$f(t) = \sum_{j=1}^{k} a_j e(\lambda_j t) \qquad [\lambda_j \text{ real}; \, e(x) = e^{2\pi i x}];$$

and, conversely, arithmetical theorems have been derived from inequalities for suitably chosen sums of this type. Thus, the classical theorems of Dirichlet and Kronecker stand in this mutual relationship to theorems about the solubility in t of the inequality $|f(t)| > \theta f^*(0)$ for a given $\theta < 1$, where $f^*(t)$ is $f(t)$ with a_j replaced by $|a_j|$. For Dirichlet's theorem the appropriate $f(t)$ have a_j real and positive, but to compensate for this restriction the solutions t can be "localized"; thus, if $\theta = \cos(2\pi/\omega)$, where $\omega > 4$, there is a solution in any given interval $\tau \leqq t \leqq \tau\omega^k$ $(\tau > 0)$. For Kronecker's theorem no restriction is placed on the a_j, but the λ_j must be supposed linearly independent (over the field of rational numbers); and there is no localization, except in special cases where the degree of linear independence can be estimated quantitatively (as, for example, when the λ_j are logarithms of primes).

By way of introducing his own point of view the author makes two comments on this situation; firstly, that the analytical theorems have equal right with the arithmetical to be considered fundamental; and, secondly, that for many purposes close localization is more useful than a strong inequality. He is thus led to formulate, as basic tools for direct application, a series of inequalities for a generalized $f(t)$ with complex λ_j. The enunciations involve only integral values of t, and for such values we may write unambiguously, after rearrangement of the terms,

$$f(t) = f_k(t) = \sum_{j=1}^{k} b_j z_j^t; \qquad |z_1| \geqq |z_2| \geqq \cdots \geqq |z_k| > 0.$$

The main results of Part I may now be stated as follows: For any $m \geqq 0$ the closed interval $[m, m+k]$ contains an integer μ such that

$$(1) \qquad |f_k(\mu)| \geqq \left(\frac{Ak}{m+k}\right)^k |z_k|^\mu |f_k(0)|,$$

and an integer ν such that

$$(2) \qquad |f_k(\nu)| \geqq \left(\frac{Bk}{m+2k}\right)^k |z_1|^\nu \min_{j=1,\cdots,k} |f_j(0)|,$$

where A, B are positive absolute constants for which possible values are $A = 1/(2e)$, $B = 1/(24e^2)$. The proof of (1) is essentially algebraical, but (2) involves some function theory as well. An example is quoted to show that these inequalities, which are developments of Littlewood's "inequality for a sum of cosines," are essentially best possible, apart from the values of A and B.

In his preface the author says that the title of his book is very pretentious, and appeals for justification to the number and variety of the applications in Part II. These applications cover such topics as: almost periodic (trigonometric) polynomials, Dirichlet series with gaps, quasi-analytic functions, boundary values of analytic functions, integral functions, differential equations, approximate solution of algebraic equations, and a variety of questions in the theory of the Riemann zeta function and the distribution of primes, with extensions to other functions defined by Dirichlet series. On the face of it this list should satisfy the most exacting critic, and it certainly leaves no doubt about the *interest* of the method. But a ready assessment of its *power* is more difficult, because its impact on existing theory is apt to take the form of new problems or modifications of old ones rather than direct improvements of known results. It is not easy to locate a conspicuous example of success where serious attempts by traditional methods have notoriously failed, though this may well be due to the very failure of such attempts. Perhaps the best way of trying to make a comparison will be to examine one particular application where it is easy to isolate the characteristic difficulty and to describe the author's method of overcoming it.

In the study of the error term in the prime number theorem, statements of the form

$$(3) \qquad \zeta(s) \neq 0 \quad \text{for} \quad \sigma > 1 - \eta(t), \quad t > t_0 \ (>0),$$

where $\zeta(s)=\zeta(\sigma+it)$ is the Riemann zeta function, are linked with statements of the form

$$\Delta(x) = O(xe^{-H(x)}) \qquad (x \geqq 1), \tag{4}$$

where $\Delta(x)$ is defined by

$$x + \Delta(x) = \psi(x) = \sum_{p^m \leqq x} \log p = \sum_{n \leqq x} \Lambda(n)$$

(p running through primes, m and n through positive integers). Here $\eta(t)$ is a positive decreasing function, $H(x)$ a positive increasing function, of sufficiently smooth behaviour. A known $\eta(t)$ leads to a possible $H(x)$ by standard routine. The general pattern of the subject suggests that the correspondence thus set up is essentially "right," and that a converse inference from $H(x)$ to $\eta(t)$ should be possible. The author shows that this is so in the special case

$$\eta(t) = c(\log t)^{-\gamma}, \qquad H(x) = C(\log x)^{1/(1+\gamma)} \qquad (0 < \gamma \leqq 1)$$

(the most interesting in our present state of knowledge), if we disregard the particular values of the positive constants c and C. The procedure is as follows. By familiar methods for "explicit formulae" it is proved that, for $\sigma > 1$, $t \geqq 0$, $\xi \geqq 1$, $h > 0$ (h integral),

$$\begin{aligned} \frac{1}{h!}\sum_{n \geqq \xi} \frac{\Lambda(n)}{n^s}\log^h\frac{n}{\xi} - \frac{\xi^{1-s}}{(s-1)^{h+1}} \\ = -\sum_{\rho} \frac{\xi^{\rho-s}}{(s-\rho)^{h+1}} + O\left(\frac{\log(t+2)}{\xi}\right), \end{aligned} \tag{5}$$

where ρ runs through the complex zeros of $\zeta(s)$, and the symbol O implies an absolute constant and an inequality valid for the full range of the relevant variables (s, ξ, h here). (This seems to be the correct form for the denominators, rather than that of Appendix V.) On the assumption that (4) is satisfied and that there is a zero $\rho^* = \sigma^* + it^*$ with t^* large and $1-\sigma^*$ small, estimates are obtained for the left-hand side of (5), for the contribution to the sum $\sum$ on the right-hand side of terms ρ outside a small neighborhood of ρ^*, and for the number of terms in the remaining finite sum $\sum'$, when ξ and h are taken fairly large and $s=\sigma+it^*$ fairly near ρ^*. The resulting estimate of $\sum'$ would yield the desired inequality $1-\sigma^* \geqq \eta(t^*)$ by proper choice of the parameters, if $\sum'$ consisted of the single term $\rho=\rho^*$. But such complete isolation is not to be expected, and there is a danger of interference from other terms. For a *given* h we have no safeguard; but by suitable use of (2) (with $b_j = 1$) it is proved that for *some h* of the appropriate order of magnitude $\sum'$ is not of significantly lower order (for the present purpose) than the single term $\rho=\rho^*$. By more traditional methods the problem may be reduced to the study of a finite sum $\sum\Re(s-\rho)^{-1}$ of positive terms presenting no interference difficulty; but without some modification this does not seem to give the precise result desired.

A similar chain of argument is used in connection with the "quasi-Riemannian hypothesis." This is the assertion that $\Theta<1$, where Θ is the upper bound of the real parts of the zeros of $\zeta(s)$. But the author prefers the equivalent statement that there is a $\vartheta<1$ such that $\zeta(s)$ has at most a finite number of zeros in $\sigma>\vartheta$, because he is concerned with necessary and sufficient conditions in terms of estimates of sums

$$\sum_{N_1\leqq p\leqq N_2} e^{-itf(p)},$$

and the parameters in these estimates are linked more naturally with ϑ than with Θ. The basic case is $f(x)=\log x$, but equivalence relations are found for other $f(x)$ also. These investigations were inspired by the hope that an independent estimate of these sums by Vinogradov's elementary method might shed new light on the existence of ϑ; but the author reports that the hope has so far proved illusory in that this method fails in the basic case $f(x)=\log x$ and gives insufficiently precise results in other cases.

A particularly interesting instance of the oblique impact of the author's method on existing theory is in the estimation of $N(\sigma, T)$, the number of zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ with $\beta\geqq\sigma$, $0<\gamma\leqq T$. There are various known estimates of the form

$$N(\sigma, T) = O(T^{\lambda(\sigma)(1-\sigma)}\log^B T) \qquad (1/2 \leqq \sigma \leqq 1) \tag{6}$$

(uniformly in σ as $T\to\infty$), where $\lambda(\sigma)$ is bounded in $1/2\leqq\sigma\leqq 1$ and B is a positive constant. The Russian school has given the name "density method" to the use of such estimates in number-theory, and "density hypothesis" to the assertion that (6) is true with $\lambda(\sigma)=2$ (and some B). In the absence of a proof, and in view of possible applications, special interest attaches to near approaches to the density hypothesis. On the Lindelöf hypothesis

$$\zeta(\alpha+it) = O(t^\epsilon) \qquad (\alpha = 1/2;\, t\to\infty), \tag{7}$$

it is known that (6) is true with $\lambda(\sigma)=2+\epsilon$ (where ϵ denotes generally an arbitrarily small positive constant). Without hypothesis (and with special reference to the neighborhood of $\sigma=1$) the author proves (6) with

$$\lambda(\sigma) = 2 + A(1-\sigma)^a \qquad (1 - b \leqq \sigma \leqq 1),$$

where A, a, b are positive constants (e.g., $A=600$, $a=1/100$). He also states that, if (7) is replaced by a similar hypothesis with a fixed α in $1/2<\alpha<1$, then (6) holds with

$$\lambda(\sigma) = 2 + 3\epsilon^{1/9} \qquad (\alpha + \epsilon^{1/3} \leqq \sigma \leqq 1).$$

But the proof is unfortunately omitted. (To reconcile the contradictory announcements in lines 1 and 22 on p. 162, read XXXVIII for XXXIX in line 22.)

In work of this kind it is perhaps inevitable that the central idea should sometimes be submerged by details. To reduce this danger the author has relegated some of the incidentals to six Appendices. Something more might be done in the same cause by a systematic use of integrals instead of sums. Thus, it would be possible to reduce to a few lines the elaborate argument used to estimate the difference on the left-hand side of (5), by writing this difference as

$$-\frac{1}{h!}\int_{\xi}^{\infty} \Delta(x) d_x\left(\frac{1}{x^s}\log^h\frac{x}{\xi}\right),$$

and using the estimate $\Delta(x)=O(xe^{-H(\xi)})$ $(x\geqq\xi)$.

In commending this stimulating book to the notice of analysts, and particularly to those interested in the analytical theory of numbers, the reviewer may perhaps be allowed two final comments on small points of wording. On pp. 131–132 it is stated that a proof of the "quasi-Riemannian hypothesis" $(\Theta<1)$ would entail essentially all the consequences of a proof of the Riemann hypothesis $(\Theta=1/2)$. This applies, of course, only to a restricted class of "consequences"; thus, the order of the error in the prime number theorem would be affected by one pair of zeros $\beta\pm i\gamma$ with $\beta>1/2$. On p. 141 reference is made to the method created by Vinogradov for the "solution of Goldbach's problem." This is a momentary lapse from more accurate descriptions elsewhere, but, since this form of words has become current in the literature, it seems opportune to point out that Goldbach's problem has not been solved. It is no disparagement of Vinogradov's outstanding achievement with the sum of three primes to state that the basic problem of two primes remains a major challenge.

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The foundation of statistics. By L. J. Savage. New York, Wiley, 1954. 16+2+294 pp. $6.00.

This book is an exposition of some of the leading ideas and tech-