Altogether we have here a very fine addition to mathematical literature.

ORRIN FRINK

On the metamathematics of algebra. By A. Robinson. (Studies in Logic and the Foundations of Mathematics.) Amsterdam, North-Holland, 1951. 9+195 pp. 18.00 fl.

The purpose of this book is to show how the methods of symbolic logic may be applied to derive new results in algebra. This is a somewhat novel idea, since usually symbolic logic has been used to strengthen the foundations, rather than to add to the superstructure, of mathematical systems. The results obtained are in the nature of metatheorems rather than theorems; they deal with entire classes of algebraic systems rather than with one system at a time.

Some typical results are the following: 1. Any theorem formulated in the restricted calculus of predicates (in terms of equality, addition, and multiplication) which is true for all commutative fields of characteristic 0, is true for all commutative fields of characteristic $p>p_0$, where p_0 is a constant depending on the theorem. 2. Any theorem of the restricted calculus of predicates which is true for all non-Archimedean ordered fields is true for all ordered fields. 3. Any theorem of the restricted calculus of predicates which is true for the field of all algebraic numbers is true for any other algebraically closed field of characteristic 0.

A feature of the method is the following: since the results require only that the algebraic systems dealt with have certain broad, general properties, they naturally suggest significant generalizations of various algebraic notions. Examples of concepts generalized are: that of algebraic number, or a number algebraic with respect to a given commutative field; the notion of the polynomial ring obtained by adjoining n indeterminates to a commutative ring; and the concept of ideal. In general, the notions that may be handled by the author's method are those capable of being formulated within the restricted predicate calculus.

A number of the results obtained are significant for symbolic logic as well as for algebra. For example, it is shown that every model of an axiomatic system formulated in the restricted calculus of predicates can be extended. Consequently such a model cannot satisfy an axiom of completeness in Hilbert's sense. Since the concept of an ordered field can be formalized within the restricted calculus of predicates while the concept of an Archimedean field does in fact possess a model which is complete in Hilbert's sense (namely the field of all real numbers), it follows that Archimedes' axiom cannot be formalized within the restricted predicate calculus. This is interesting from the logical viewpoint.

The first chapter is a general introduction, summarizing the work. Chapter II, Construction of a formal language, describes the symbolism, rules of formation and procedure, and semantical interpretation of the object language. The resulting system is similar to the restricted predicate calculus of Hilbert. This chapter gives the logical machinery. The third chapter, which deals with the relation between deductive concepts on the one hand, and semantic or descriptive concepts on the other hand, contains also an important completeness theorem. One formulation of this theorem is: For every consistent set of statements K in a language L there exists a structure M such that all the statements of K hold in M, for some correspondence C. A consequence is a theorem of Lindenbaum, to the effect that every consistent set of statements K in a language L is included in a complete set K^*.

In the fourth chapter algebraic notions are introduced. Axioms are given for groups, rings, and fields in the logical notation used, and the notion of characteristic of a field and the Archimedean axiom are formalized. In Chapter V metamathematical theorems on algebraic fields such as those mentioned above are proved.

Chapter VI is more general, and deals with the broad notion of algebraic systems and algebras of axioms, rather than with special notions such as group, ring, and field. This chapter contains results on extensions of algebraic systems. The seventh and eighth chapters, on polynomials and algebraic predicates, deal with generalizations of the notions of polynomial and algebraic equation. The ninth chapter is about so-called convex systems. A system of axioms is convex, broadly speaking, if the intersection of any number of models of the system is a model. The last two chapters, on ideals and pre-ideals, are concerned with generalizations of the notion of ideal and of algebraic variety.

The author emphasizes that he has given more or less random samples of the type of result obtainable by his method. Many directions for further generalization are open. He has made a clear and compelling case for his principal contention, namely, that symbolic logic, in addition to its usefulness in investigating the foundations of algebra, is also a practical tool for obtaining new results in algebra of a very general sort, and results of a type not readily obtainable by other methods.

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