GILBERT AMES BLISS
1876–1951

The contributions made by Gilbert Ames Bliss to mathematics and to educational and scientific activities in the United States were many and varied. The following is a quite inadequate summary of his life and work, with special reference to his mathematical activities.

Gilbert Bliss was born in Chicago on May 9, 1876, and died in Ingalls Memorial Hospital in Harvey, Illinois, on May 8, 1951. In 1893, one year after the University of Chicago first opened its doors to students, he enrolled there as a student, and was awarded the degrees of B.S. in 1897, M.S. in 1898, and Ph.D. in 1900. He studied at the University of Göttingen during the year 1902–1903. His first teaching was done as a substitute for a member of the staff of Kalamazoo College for several weeks during the year 1898–1899. He was an instructor in mathematics at the University of Minnesota 1900–1902, an associate at Chicago 1903–1904, assistant professor at the University of Missouri 1904–1905, preceptor (=assistant professor) at Princeton 1905–1908, and at Chicago was associate professor 1908–1913, professor 1913–1933, Martin A. Ryerson Distinguished Service Professor 1933–1941, and chairman of the department 1927–1941. He retired from active service in 1941. In various summer or autumn terms from 1906 to 1911 he gave courses at Wisconsin, Chicago, Princeton, and Harvard.

During his student days Bliss first fell under the influence of F. R. Moulton, then a young assistant who was teaching at Chicago, and his first published paper was entitled *The motion of a heavenly body in a resisting medium.* The fellowship in astronomy for which he applied was not granted, and he eventually decided to devote himself to pure mathematics. His thesis for the M.S. was entitled *The geodesic lines on the anchor ring*, and his thesis for the Ph.D. bore the same title. The first developed explicit formulas for the geodesics in terms of elliptic integrals and discussed some of their properties. In the Ph.D. dissertation (**D3**)[1] it was shown that the points on the inner equator of the anchor ring are of the first kind (in the classification due to Mangoldt), i.e., each geodesic passing through such a point contains no conjugate point, while all other points on the anchor ring are of the second kind, i.e., not of the first kind. Previous investigations by Jacobi and by Mangoldt had shown that on a surface

[1] The letters in bold face refer to the sections of the bibliography at the end of this article, and the numbers refer to the individual items.

of negative curvature all points are of the first kind, while on a closed surface of positive curvature all points are of the second kind.

During his time as a graduate student Bliss made a copy of Bolza's record of Weierstrass' 1879 course on the calculus of variations. This record together with Bolza's influence undoubtedly helped fix in Bliss' mind the interest in the subject which dominated his research. During his stay at Minnesota he studied Kneser's book on the subject, which was the first printed exposition of Weierstrass' ideas, and published a paper (**D4**) on the second variation and sufficient conditions for a minimum when one end point is variable. A second paper (**D5**), taking up the case when both end points are variable, was written during his stay in Göttingen. In both these papers, as in his thesis, geometric considerations play a prominent role. There are frequent evidences of his continuing interest in geometry. In his papers **D11, 14, 27**, he discussed "Finsler" geometry of two dimensions, where the arc length is given by an integral of the form

$$\int f(x, y, \tau)(x'^2 + y'^2)^{1/2} dt, \quad \tan \tau = \frac{y'}{x'} \cdot$$

Finsler's thesis treating the n-dimensional case did not appear until 1918. The tensor analysis of such spaces was developed still later by Élie Cartan. Bliss kept in touch with these ideas, and developments of them appeared in the dissertations of Taylor (**G18**), Johnson (**D28**), Householder (**G47**), and Stokes (**G51**), which he suggested and supervised. During the years 1908–1910, immediately following the death of Maschke, he lectured on geometry at Chicago, but when Bolza returned to Germany in 1910, Bliss seemed glad to make analysis again the focus of his teaching.

A number of papers applying the methods of Weierstrass to a variety of problems in calculus of variations appeared from Bliss' pen during the years before the first world war. Some of these were written in collaboration with Max Mason (**D12, 17, 19**) and one was a joint paper with A. L. Underhill (**D25**).

An idea which has become basic for much subsequent work in the calculus of variations appeared in the paper **D30**. This is the consideration of the minimum properties of the second variation. It makes possible simple proofs of the necessity of the Jacobi condition, and of the analogous condition for the more general problem of Bolza, and is a guiding principle in the construction of sufficiency proofs. It led to the paper **D44** on the transformation of Clebsch, which expresses the second variation in the form

$$I_2(\eta) = \int_{x_1}^{x_2} F_{y_i' y_k'}(\eta_i' - \pi_i)(\eta_k' - \pi_k)dx.$$

This formula is valid when there exists a conjugate system of accessory extremals whose determinant does not vanish on the interval $[x_1, x_2]$ and the vector function π is suitably defined in terms of the admissible variation η (vanishing at x_1 and x_2) and of the given conjugate system. It is derived very simply with the help of the theory of fields of extremals and the formula of Weierstrass. A simple direct proof is also given. The methods of this paper are basic for the various sufficiency proofs subsequently given for the problems of Lagrange, Mayer, and Bolza.

During the years 1916–1946 Bliss devoted much time to improving and extending the theories of the problems of Lagrange, Mayer, and Bolza. In this he had the cooperation of several of his students, notably Hestenes. The results appeared in definitive form in 1946 in the second part of his "Lectures" (**A6**), where the problem of Bolza is treated in detail. If one compares this with earlier expositions of these general problems—for example, of the problem of Lagrange in Bolza's *Vorlesungen*, or in Bliss' paper **D49**—one is immediately impressed with the greater scope of the theory, due to weakening of hypotheses both for necessary conditions and for sufficient conditions, and with the simplifications obtained in the proofs. The first part of the "Lectures" contains an unusually clear presentation of the theory of the calculus of variations for cases when there are no side conditions.

Although Bliss published only one paper on multiple integrals in the calculus of variations (**D59**) he began to discuss the subject in courses and seminars in the 1920's, and continued at frequent intervals up to the summer of 1942. Coral, Courant, McShane, Radó, and Smiley were among those from outside the University of Chicago who participated in some of these seminars. Bliss expounded the subject of multiple integrals in mimeographed lecture notes (**B2, 3**), where various improvements in the theory may be found. Some of his ideas were developed in the doctoral dissertations of Simmons (**G19**), Coral (**G35**), Raab (**G39**), Cosby (**G41**), Nordhaus (**G50**), and Landers (**G52**). One of the outstanding problems in this domain is to find conditions ensuring the existence of a field suitable for use in a sufficiency proof.

During the 1920's Bliss also lectured on boundary value problems associated with the calculus of variations, and on applications to quantum mechanics and relativity. Some of his work on boundary value problems appeared in the papers **D45**, **46**, **50**, **58**. For the sake of the greatest symmetry and generality it is convenient to study differential systems of the form

$$\frac{dy}{dt} = [A(x) + \lambda B(x)]y, \qquad My(a) + Ny(b) = 0. \tag{1}$$

(Here capital letters are used to denote square matrices of order n, y and z denote matrices of one column, and the transpose of A, for example, is denoted by A'.) The adjoint system to (1) is defined to be

$$\frac{dz}{dt} = - z'[A + \lambda B], \qquad z'(a)P + z'(b)Q = 0,$$

where $MP - NQ = 0$. The system (1) is said to be self-adjoint if it is equivalent to its adjoint under a transformation $z = T(x)y$ where the matrix T is nonsingular on the interval $[a, b]$. In the paper **D58**, a self-adjoint system (1) was defined to be *definite* in case: (a) the matrix $T'B$ is symmetric and positive semidefinite, and (b) the system (1) has no nontrivial solutions for which $By = 0$. The boundary value problems arising from problems of Bolza satisfying certain mild restrictions are of this type. The definition of "definitely self-adjoint" given in the earlier paper **D46** excluded many such problems. However, in both papers properties of the system are derived which are like those which hold for the Fredholm equation with real symmetric kernel. Student theses related to the subject of boundary value problems include those of Miss Stark (**G21**), Bamforth (**G22**), Cope (**G23**), Hickson (**G26**), Miss Jackson (**G27**), Hu (**G38**), and Miss Wiggin (**G43**).

The inverse problem of the calculus of variations drew some attention from Bliss, although it was not in the main line of his interest. (See the paper **D15**.) Among his students Davis (**G24**), LaPaz (**G29**), and Moscovitch (**G46**) contributed to the discussion of the problem.

Calculus of variations theory requires the use of existence theorems for implicit functions and differential equations which yield more information than those commonly given in the textbooks and treatises on analysis. Hence Bliss was led to write a number of papers on the subject (**D6**, **9**, **20**, **33**), and his Colloquium lectures delivered before the American Mathematical Society in 1909 (**A1**) were devoted to these and related topics. The analysis of singular points for transformations of the plane was one of the topics treated in the Colloquium. This work grew out of special cases arising in the calculus of variations. The dissertation of Lovitt (**G6**) treated some cases for transformations of three-space. The transformations of the plane considered in the Colloquium are real analytic ones, and real branches of plane analytic curves play a prominent role. This led to the two papers on the factoring of power series (**D18, 21**), and later to the study of algebraic curves (**D41, 42, A4**). At the date of these studies the geometric proofs for the theorems on the reduction of the singularities of algebraic curves were quite unsatisfactory. In recent years modern algebra has provided the means for a very abstract and general treatment of the theory of algebraic curves. Bliss gave a clear treatment from the viewpoint of analysis of the case when the base field is the complex number field. This phase of his work indicates that he was not always interested in the maximum abstractness, although he always sought for simplicity, clarity, and comprehensiveness in his mathematical exposition.

Upon the urging of Veblen in 1918 Bliss went to Aberdeen as a scientific expert in the Range Firing Section. His work on the differential corrections of trajectories (**D35, 36, 37**) made possible enormous savings in the time required to compute the range corrections which are necessary in allowing for the rotation of the earth, effects of wind, and variations in the density of the air, powder charge, etc. His methods continued to be used during World War II. While the advent of high speed computing machines has led to some changes in methods, the basic theoretical ideas remain the same. These basic ideas were expounded in two papers in Trans. Amer. Math. Soc. (**D39, 40**), and later in a book (**A5**). Bliss' interest in functional analysis was not confined to this period. In earlier years the dissertations of Fischer (**D5**), Lamson (**G10**), Le Stourgeon (**G11**), and Barnett (**G13**) were related to this field.

Bliss was known the world around as one of the leading authorities on the calculus of variations, although he did not contribute to the new directions of study opened up by Tonelli and by Morse. He was primarily interested in mathematical research, but he heeded also the call of other duties. His broad interest in mathematics was evidenced, for example, by his regular attendance at the meetings of the Mathematical Club of the University of Chicago, where he contributed comments and questions on a wide variety of topics. He felt the duty of exposition, and wrote the first of the series of Carus Monographs, for which he also served as a member of the editorial committee. In mathematical publication he took a middle ground between those who pour forth undigested ideas and those who insist on excessive polishing. In 1925 he was the recipient of the first award of the Chauvenet Prize by the Mathematical Association of America for his paper *Algebraic functions and their divisors* (**D43**).

Bliss was an associate editor of the Annals of Mathematics from 1906 to 1908, and of the Transactions of the American Mathematical Society from 1909 to 1916. He was chairman of the Editorial Committee of the University of Chicago Science Series from 1929 until his retirement. From 1924 to 1936 he served on the Fellowship Board of the National Research Council. He was a trustee of the Teachers Insurance and Annuity Association for several years. A wide variety of organizations called on him to make informal addresses, and as a result he gave quite a number of such talks on mathematics and related topics.

He was elected to the National Academy of Sciences in 1916, to the American Philosophical Society in 1926, and was made a Fellow of the American Academy of Arts and Sciences in 1935. During his term (1921 and 1922) as president of the American Mathematical Society, he conducted a campaign for new members, in cooperation with E. R. Hedrick, Chairman of the Membership Committee. As a result the membership of the Society increased from 770 to 1127 during this biennium. In 1930 he was Vice President and Chairman of Section A of the American Association for the Advancement of Science. He was also a member of the Mathematical Association of America, the Illinois Academy of Science, the London Mathematical Society, the Deutsche Mathematische Verein, and the Circolo Matematico di Palermo. In 1935 the University of Wisconsin awarded him the honorary degree of Doctor of Science.

Bliss played a central role in the planning of Eckhart Hall, which houses the Department of Mathematics at Chicago, and he closely supervised its erection. It is very largely due to his care and insight that this building fills so well the needs of the Department, and it will stand as a beautiful and enduring memorial to his efforts. He accumulated a rather substantial personal library of mathematical books and periodicals, which he presented to the Department of Mathematics in 1950 for the use of the staff. This "Gilbert Ames Bliss Library" is now housed in the Faculty Conference Room in Eckhart Hall. It includes an unusually complete collection of works on the calculus of variations.

Bliss believed wholeheartedly in the importance of teaching as an accompaniment of mathematical research. All those who experienced the stimulation of his courses retain a high admiration for him as very nearly the ideal scientific teacher. His judgment was highly regarded, and his counsel was sought from many quarters, by other universities as well as his own, and by other organizations besides the mathematical ones. He was conservative in tendency, and was not one to rush into new domains of mathematics or into new projects in war time. In his view fundamental mathematical research and teaching were activities too important to be interrupted except for very strong reasons. Attempts to restrict student enrollments by high admission requirements seemed to him unwise, since some poorly trained students develop real intellectual power, and some brilliant ones gradually fade. With respect to the value of the Ph.D. training, he stated:

"The real purpose of graduate work in mathematics, or in any other subject, is to train the student to recognize what men call the truth, and to give him what is usually his first experience in searching out the truth in some special field and recording his impressions. Such a training is invaluable for teaching, or business, or whatever activity may claim the student's future interest."

Gilbert Bliss was married to Helen Hurd in 1912. Their children are Elizabeth (Mrs. Russell Wiles), born in 1914, and Ames, born in 1918. His wife was stricken and died in the influenza epidemic of December, 1918. In 1920 he married Olive Hunter, who survives him. The Blisses had a summer home in Flossmoor, Illinois, for many years, and beginning in 1931 they made their year round residence in Flossmoor, where Gilbert was at one time a member of the Village Board of Trustees and head of the Police Commission. They were exceptionally friendly in manner and in spirit, with a high sense of humor, and enjoyed entertaining students, faculty colleagues, and friends in their home. Golf at one of the nearby country clubs was a favorite recreation until declining health forced its discontinuance. Bliss was interested in competitive sports throughout his life, and in his earlier years participated actively in bicycle racing, tennis, and racquets.

His influence on mathematics and mathematicians was widespread and deep, and his contributions will be long remembered.

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16. Lawrence Murray Graves, *The derivative as independent function in the calculus of variations.*
17. Jewell Constance Hughes, *A problem in the calculus of variations in which one end-point is variable on a one-parameter family of curves.*
18. James Henry Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas.*

1925

19. Harvey Alexander Simmons, *The first and second variations of a double integral for the case of variable limits.*

1926

20. Irvine Rudsdale Pounder, *A method of successive approximations for a partial differential equation of hyperbolic type.*
21. Marion Elizabeth Stark, *A self-adjoint boundary value problem associated with a problem of the calculus of variations.*

1927

22. Frederic Richard Bamforth, *A classification of boundary value problems for a system of ordinary differential equations of the second order.*
23. Thomas Freeman Cope, *An analogue of Jacobi's condition for the problem of Mayer with variable end points.*
24. David Roy Davis, *The inverse problem of the calculus of variations in higher space.*

25. Ernest Lloyd Mackie (with L. M. Graves), *The Jacobi condition for a problem of Mayer with variable end points.*

1928

26. Arthur Owen Hickson, *An application of the calculus of variations to boundary-value problems.*

1928

27. Rosa Lea Jackson, *The boundary-value problem of the second variation for parametric problems in the calculus of variations.*
28. Marie Mathilda Johnson, *Tensors of the calculus of variations.*
29. Lincoln La Paz, *An inverse problem of the calculus of variations.*

1930

30. Moffatt Grier Boyce (with L. M. Graves), *An envelope theorem and necessary conditions for a problem of Mayer with variable end-points.*
31. William Larkin Duren, Jr., *The development of sufficient conditions in the calculus of variations.*
32. Aline Huke, *An historical and critical study of the fundamental lemma in the calculus of variations.*
33. Edward James McShane (with L. M. Graves), *Semi-continuity in the calculus of variations, and absolute minima for isoperimetric problems.*
34. Frank Lynwood Wren, *A new theory of parametric problems in the calculus of variations.*

1931

35. Max Coral, *The Euler-Lagrange multiplier rule for double integrals.*
36. Ralph Grafton Sanger, *Functions of lines and the calculus of variations.*

1932

37. Magnus Rudolph Hestenes, *Sufficient conditions for the general problem of Mayer with variable end-points.*
38. Kuen-Sen Hu (with L. M. Graves), *The problem of Bolza and its accessory boundary value problem.*
39. Albert William Raab (with L. M. Graves), *Jacobi's condition for multiple integral problems of the calculus of variations.*

1933

40. Julia Wells Bower, *The problem of Lagrange with finite side conditions.*

1934

41. Byron Cosby II, *Fields for multiple integrals in the calculus of variations.*
42. Van Bauman Teach (died before taking examination), *The Hamilton-Jacobi theory for the problem of Lagrange in parametric form.*

1936

43. Evelyn Prescott Wiggin (with W. T. Reid), *A boundary value problem of the calculus of variations.*
44. Ross Harvey Bardell, *The inequalities of Morse for a parametric problem of the calculus of variations.*

1937

45. Carl Herbert Denbow (with L. M. Graves and W. T. Reid), *A generalized form of the problem of Bolza.*
46. Nathan Abraham Moscovitch, *Studies of the inverse problem of the calculus of variations.*
47. Alston Scott Householder, *The dependence of a focal point upon curvature in the calculus of variations.*

48. Frederick Albert Valentine (with L. M. Graves), *The problem of Lagrange with differential inequalities as added side conditions.*

1938

49. Carroll Parker Brady (with M. R. Hestenes), *The minimum of a function of integrals in the calculus of variations.*

1939

50. Edward Alfred Nordhaus (with M. R. Hestenes), *The problem of Bolza for double integrals in the calculus of variations.*

51. Ellen Clayton Stokes, *Applications of the covariant derivative of Cartan in the calculus of variations.*

52. Aubrey Wilfred Landers, Jr. (with M. R. Hestenes), *Invariant multiple integrals in the calculus of variations.*

53. Mary Kenny Landers (with M. R. Hestenes), *The Hamilton-Jacobi theory for problems of Bolza and Mayer.*

L. M. GRAVES