with a brief account (8 pages) of finite linear spaces. There are no exercises.

The book treats with clarity and precision an astonishing amount of material, and is a very welcome addition to the literature of the subject.

LEONARD M. BLUMENTHAL

The location of critical points of analytic and harmonic functions. By J. L. Walsh. (American Mathematical Society Colloquium Publications, vol. 34.) New York, American Mathematical Society, 1950. 8+384 pp. $6.00.

As the title indicates, the book is concerned with the critical points of analytic functions $f(z)$ of the single complex variable $z=x+iy$ and of harmonic functions $u(x, y)$ of the two real variables x and y. As is well known, a critical point of $f(z)$ means a zero of its derivative $f'(z)$, and a critical point of $u(x, y)$ means a point where both partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ vanish. The former are the points where the map by $w=f(z)$ fails to be conformal and are the multiple points of the curves $|f(z)|=\text{const.}$ and $\arg f(z)=\text{const.}$ The latter are the equilibrium points in the force field having $u(x, y)$ as force potential and are the stagnation points in the flow field having $u(x, y)$ as velocity potential. Thus the subject matter of the book is one of considerable importance in both pure and applied mathematics.

In this book the analytic functions considered are largely polynomials, rational functions, and certain periodic, entire, and meromorphic functions. The harmonic functions considered are largely Green's functions, harmonic measures, and various linear combinations of them. The interest in these functions centers about the approximate location of their critical points. The approximation is in the sense of determining minimal regions in which lie all the critical points or maximal regions in which lies no critical point.

This book not only has a unity of subject matter, but it also has very nearly a unity of method. The method is based upon the observation that, with $z-a=re^{i\theta}$ and thus $(\bar{z}-\bar{a})^{-1}=r^{-1}e^{i\theta}$, the vector $(\bar{z}-\bar{a})^{-1}$ has the direction of the line segment from the point a to the point z and a magnitude equal to the reciprocal of the length of this line segment. Accordingly, the vector $m(\bar{z}-\bar{a})^{-1}$ may be regarded as force with which a particle of mass m repels a unit particle at z; the sum $F(z)=\sum m_k(\bar{z}-\bar{a}_k)^{-1}$ may be regarded as the resultant force upon a unit particle at z due to the system of discrete masses m_k at a_k, and the integral $J(z)=\int[\bar{z}-\bar{a}(t)]^{-1}dm(t)$ may be regarded as the resultant force at z due a continuous spread of matter. Now, it turns out that the critical points of the various functions studied in this book are the zeros of functional forms $F(z)$ and $J(z)$. Hence, the critical points in question may be sought as positions of equilibrium in the above fields of force.

This method of studying critical points was introduced by Gauss during the first third of the nineteenth century. (The exact date is in doubt, some books including Walsh's taking it as 1816, but others including the reviewer's taking it as nearer 1836.) Gauss identified the critical points of polynomials with the equilibrium points in a field of positive discrete particles. In generalization, Bôcher identified the critical points of rational functions with the equilibrium points in a field of both positive and negative discrete particles. But no one has carried this method quite as far and as skillfully as Walsh. On reading this book, those who have not seen Walsh's earlier work will be amazed at the breadth, depth, and beauty of the results which he is able to derive by this method of force fields.

Walsh's book is divided into nine chapters. The first five cover the polynomials and rational functions. The sixth has to do with certain entire, meromorphic, and periodic functions. The final three chapters deal with harmonic functions.

Chapter I, entitled *Fundamental results*, opens with the definition of terms such as Jordan arc, curve, and configuration, and with a statement of certain theorems needed subsequently, such as the Principle of Argument and Harnack's theorm. The Principle of Argument is then used to derive Rouché's Theorem which in turn is used to derive Hurwitz' Theorem and the theorem on the continuity of the zeros of a polynomial as functions of the coefficients of the polynomial.

After these preliminaries, Chapter I proceeds to the subject matter proper by establishing the four basic theorems due to Gauss, Lucas, Jensen, and Walsh. The *theorem of Gauss* states that every critical point of a polynomial $P(z)=\prod_{j=1}^{p}(z-z_j)^{m_j}$ lies at a position of equilibrium in the field of force due to particles of mass m_j at z_j repelling a unit particle at z according to the inverse distance law. From this theorem there follows at once *Lucas' Theorem* that every critical point of $P(z)$ lies in the smallest convex polygon enclosing the zeros z_j of $P(z)$. When $P(z)$ is real, the consideration of the resultant of the forces due to the equal particles at the two conjugate zeros z_j and $\bar{z}_j$ leads to *Jensen's Theorem* that every non-real critical point of real polynomial lies in at least one of the so-called Jensen circles of $P(z)$. The Jensen circles of $P(z)$ are those whose diameters are the line-segments joining the pairs of conjugate imaginary zeros

z_j and $\bar{z}_j$ of $P(z)$. Finally, there is *Walsh's Theorem* that, if all the z_j, $1 \leqq j \leqq q$ $(q < p)$, lie in or on a circle C_1 with center at α_1 and radius r_1 and if all the z_j, $q+1 \leqq j \leqq p$, lie in or on a circle C_2 with center α_2 and radius r_2, then all the critical points of $P(z)$ lie in or on C_1, C_2, and a third circle C_3, with center $(n_1\alpha_2 + n_2\alpha_1)/(n_1+n_2)$ and radius $(n_1r_2 + n_2r_1)/(n_1+n_2)$, where $n_1 = m_1 + \cdots + m_q$ and $n_2 = m_{q+1} + \cdots + m_p$. This theorem is proved by showing that at any point z outside C_1 and C_2 the resultant force due to the masses m_j at the z_j is the same as that due to a particle of mass n_1 at a suitable point in C_1 and a particle of mass n_2 at a suitable point in C_2.

The last topics discussed in Chapter I are the elementary properties of the lemniscates $|P(z)| = \text{const.}$ and their orthogonal trajectories $\arg P(z) = \text{const.}$

Chapter II, entitled *Real polynomials*, includes also certain nonreal polynomials which possess a real polynomial factor. The chapter opens with some theorems that sharpen Rolle's Theorem as applied to polynomials all of whose zeros are real. Then the author returns to Jensen's Theorem. After restating it in an equivalent form involving equilateral hyperbolas instead of circles, he adds the following result of his own: Let K be a configuration comprised of the segment (a, b) of the real axis $(P(a)P(b) \neq 0)$ and the closed interiors of the Jensen circles that intersect this segment; then, if k z_j lie in K, then $k-1$, k, or $k+1$ critical points lie in K. This result is deduced by a study of the force field and use of the Principle of Argument. Further theorems of the Jensen type are also developed for polynomials of the forms $P(z) = (z^2+1)^k p_1(z)$ where all the zeros of $p_1(z)$ are real, $P(z) = p_1(z)p_2(z)$ where $p_1(z)$ is real and $p_2(z)$ has all its zeros in the upper half-plane, and $P(z) = p_1(z)p_2(z)$ where the zeros of $p_1(z)$ are symmetric in the origin and lie in a double sector R of opening less than $\pi/2$ and the zeros of $p_2(z)$ lie in one of the sectors complementary to R.

In Chapter II, also, Walsh introduces the novel idea of a W-curve for a given polynomial $P(z)$ whose zeros z_j are subject to some prescribed symmetry S. This curve is the locus of the critical points of $P(z)$ when the "multiplicities" of the z_j are allowed continuously to take on all possible values consistent with S. Thus, for $P(z) = (z^2+1)^k(z-a)^m$ with $a > 0$, the W-curve is the line-segment $0 \leqq x \leqq a$ of the real axis plus that arc of the circle: $a(x^2+y^2-1)+2x = 0$ which lies inside the unit circle. More generally, if the zeros z_j of $P(z)$ fall into two groups G_1 and G_2 each satisfying S, the corresponding W-curve consists of G_1 and G_2 and the set of all points Q at which the force due to G_1 has a direction opposite to the force due to G_2; for, by suitable assignment of multiplicites to the z_j in G_1 and G_2, point Q can be made a point of equilibrium and hence a critical point. This fact leads to the following broad generalization of Lucas' Theorem: If T denotes the set of the W-curves corresponding to all possible groups G_1 and G_2 for the given z_j and the given symmetry S, then no point Q is a critical point of $P(z)$ if it can be joined to the point at infinity by a Jordan arc which does not intersect T.

Chapter III entitled *Polynomials, continued*, falls into three parts. First, there are developed three analogs and extensions of Walsh's Theorem: (1) the analog when the exterior instead of the interior of circle C_2 is used; (2) the analog when two half-planes are used instead of the circles C_1 and C_2; (3) the generalization when, instead of just two circles, there are n circles C_j having a common external (finite or infinite) center of similitude. Secondly, some results analogous to those in Chapter II are developed for polynomials $P(z)$ symmetric in the origin or, more generally, having a k-fold symmetry in the origin. These new results are derived by applying the previous results to the polynomial $p(w)=[P(z)]^k$ where $w=z^k$. Thirdly, a brief study is made regarding the zeros of higher derivatives of a polynomial $P(z)$. By iteration of the results for the first derivative, Lucas' Theorem is found to be valid also for the kth derivative, but in Jensen's Theorem the Jensen circles must be replaced by the ellipses whose minor axes are the line segments joining the pairs of conjugate imaginary zeros z_j and $\bar{z}_j$ of $P(z)$ and whose eccentricity is $(1-k^{-1})^{1/2}$.

Chapter IV, entitled *Rational functions*, opens with Bôcher's extension of Gauss' Theorem to fields containing both positive and negative particles and thus to rational functions $R(z)$. This force field is invariant under linear transformation provided that a particle of suitable mass is placed at the image of the point at infinity if the latter is a zero or pole of $R(z)$. From the fact that the force due to a pair of particles of masses m and $-m$ is directed along the circle through the particles towards the negative mass, one deduces the following (*Bôcher's Theorem*): If C_1 and C_2 are two disjoint circular regions which contain respectively all the zeros and all the poles of a rational function $R(z)$, then C_1 and C_2 also contain all the critical points of $R(z)$.

This theorem, as well as Walsh's Theorem, are given the following beautiful generalization (*Walsh's Cross-Ratio Theorem*): Let C_1, C_2, and C_3 be closed circular regions with C_3 disjoint from C_1+C_2. Let k zeros of a rational function $R(z)$ of degree $n>k$ have C_1 as their common locus, the remaining zeros have C_2 as their common locus and all the poles have C_3 as their common locus. Then the locus of the critical points $R(z)$ consists of C_1 if $k>1$, C_2 if $n-k>1$, the interior points of C_3, and a fourth circular region C_4. This fourth circular region is locus of the point z_4 défined by the cross-ratio $(z_1, z_2, z_3, z_4) = n/k$, as z_1, z_2, and z_3 vary independently over C_1, C_2, and C_3. Walsh's proof of his theorem is quite geometrical.

Chapter IV closes with a statement of *Marden's Theorem.* This is a generalization of Walsh's Cross-Ratio Theorem to rational functions of the form $R(z) = [f_1(z)f_2(z) \cdots f_q(z)]/[f_{q+1}(z)f_{q+2}(z) \cdots f_p(z)]$, where $f_k(z)$ is a polynomial whose zeros all lie in a given circular region C_k. Any critical point of $R(z)$ which lies outside all the given regions C_k lies in a point-set bounded by some or all of the ovals of a certain curve whose equation has the form $A(x^2+y^2)^{p-1}+\phi(x, y)=0$ where in general $A\neq 0$ and where $\phi(x, y)$ is a polynomial in x and y of combined degree less than $2(p-1)$.

Chapter V, entitled *Rational functions with symmetry*, is the longest chapter in the book. It includes some of Walsh's more recent results.

The first symmetry studied is that in the real axis. Somewhat parallel to the earlier sections of Chapter III, the earlier sections of Chapter V deal with rational functions, all of whose zeros are real and with the extension of Jensen's theorem to arbitrary real rational functions and especially to real rational functions whose only poles are $\pm i$.

The second symmetry, that in the unit circle C, is even more interesting as it involves hyperbolic, non-euclidean (abbreviated: NE) geometry. One example is that of a rational function $R(z)$ whose zeros and poles are interlaced on C. For such an $R(z)$ a Lucas-type theorem is shown to hold: namely, that all the critical points within C lie in the NE convex polygon whose sides are the NE lines for C joining consecutive zeros and those joining consecutive poles.

Another example of symmetry in the unit circle is the rational function of the form

$$R(z) = \lambda \prod_{k=1}^{m} \left(\frac{z-\alpha_k}{1-\bar{\alpha}_k z}\right), \qquad |\alpha_k| < 1, \ |\lambda| = 1.$$

For this $R(z)$ a Lucas type of theorem is proved to the effect that all the critical points of $R(z)$ in C lie in the smallest closed NE polygon Π enclosing the α_k. Also, for this same $R(z)$ if real, a Jensen-type theorem is established with the ordinary Jensen circles replaced by circles having as NE diameters the NE line segments joining the pairs of conjugate imaginary α_j. Likewise, for a quotient of two such $R(z)$, a Bôcher type theorem is established; namely, that under certain conditions the critical points in C of the quotient lie within two disjoint NE polygons.

As to the method of proving the results just mentioned, that used for the Lucas-type theorem is typical. A point z_0 is chosen in C outside Π and, through z_0, a NE line L is drawn not containing any point in or on Π. Now region C is mapped upon itself so that z_0 goes into the origin O, L into the real axis, and the α_j into the points α_j' of the upper half plane. At O the force due to unit particles at α_j' and $1/\alpha_j'$ will for all j have a component in the negative direction of the axis of imaginaries and hence O cannot be an equilibrium point. Hence, no point z_0 in C outside Π can be a critical point of $R(z)$.

In addition to rational functions symmetric in the real axis and the unit circle, Chapter V includes a treatment of those symmetric in the origin, those skew-symmetric in the origin, and those symmetric in z and $1/z$. The results are too numerous to mention.

Chapter VI, entitled *Analytic functions*, falls into three parts.

The first part is concerned with the extension of Lucas', Jensen's, and Walsh's Theorems to entire functions $f(z)=\prod_1^\infty (1-z/\alpha_k)$ of genre zero and of Bôcher's Theorem to quotients of two entire functions of genre zero. Also, in the first part, infinite Blaschke products are considered, and a Bôcher type theorem as applied to a double sector is developed for

$$\Psi(z) = \prod_{m=-\infty}^{\infty} \prod_{k=1}^{n} \frac{(z-\rho^m\alpha_k)(1-\rho^m\beta_k)}{(1-\rho^m\alpha_k)(z-\rho^m\beta_k)},$$

a function with a multiplicative period of ρ.

In the second part of Chapter VI, conformal mapping is employed to extend the previous results on rational functions over to more general types of functions. For example, let $f(w)$, an analytic function of $w=u+iv$ with a period of $2\pi i$, have a period strip that has no singularity except a pole at its right-hand end point. Then, by mapping the period strip upon a half-plane and by using Lucas' Theorem, one may prove that if all the zeros of $f(w)$ satisfy the inequality $u\leqq c$, or $a\leqq v\leqq b$ with $b-a<2\pi$, so do also the critical points of $f(w)$. In the same manner, a Bôcher-type theorem is derived for meromorphic functions that are simply or doubly periodic.

In the third part of Chapter VI, the possibility of the use of Cauchy's Integral Formula in the investigation of critical points is explored. If $f(z)$ is analytic in a region R bounded by two disjoint Jordan configurations C_1 and C_2, and if $|f(z)|=M_1$ on C_1 and $|f(z)|=M_2$ on C_2 where M_1 and M_2 are constants with $0<M_1<M_2$, then

the Cauchy Integral Formula leads to the representation

$$f'(z)/f(z) = (2\pi i)^{-1}\left\{\int_{C_1}(z-t)^{-1}d\mu_1 + \int_{C_2}(z-t)^{-1}d\mu_2\right\},$$

where $d\mu_k(t) = -d\,[\arg f(z)]$, a quantity positive on C_1 and negative on C_2 with $\int_{C_1} d\mu_1 + \int_{C_2} d\mu_2 = 0$. The critical points are therefore the equilibrium points in the field of force due to a spread of positive matter on C_1 and of negative matter on C_2, with the total mass zero. Consequently, the Bôcher-type theorem is valid; namely, if C_1 and C_2 lie respectively in the disjoint circular regions K_1 and K_2, all the critical points of $f(z)$ also lie in K_1 and K_2.

Chapter VII, entitled *Green's functions*, is devoted largely to the Green's function $G(x, y)$, with pole at infinity corresponding to an infinite simply-connected region R bounded by a finite Jordan configuration B. For such functions is developed the representation

$$G(x_0, y_0) = \int_B (\log r)d\sigma + g,$$

where (x_0, y_0) is a point in R, r is the distance of a variable point on B from (x_0, y_0), g is a constant, and $d\sigma > 0$ with $0 \leqq \sigma \leqq 1$. On introducing the harmonic conjugate $H(x, y)$ to $G(x, y)$ and defining $F(z) = G(x, y) + iH(x, y)$, one may write

$$F'(z) = \int_B (z-t)^{-1}d\sigma(t).$$

Thus, the critical points are the points of equilibrium in the field of force due to a spread of positive matter on B. Therefore, theorems of the Lucas, Jensen and Walsh types are valid. In particular, the Walsh type theorem reads: If $B = B_1 + B_2$ where B_1 and B_2, two disjoint Jordan configurations, lie respectively in the circles C_1: $|z-\alpha_1| = r_1$ and C_2: $|z-\alpha_2| = r_2$, then the critical points lie in C_1, C_2, and a third circle C_3: $|z-(m_2\alpha_1+m_1\alpha_2)| = m_2r_1+m_1r_2$ where $m_1 = \int_{B_1}d\sigma$, $m_2 = \int_{B_2}d\sigma$, and $m_1+m_2 = 1$. Furthermore, if B_1 and B_2 consist respectively of k_1 and k_2 components and if the closed regions C_1, C_2, and C_3 are disjoint, then C_1, C_2, and C_3 contain respectively k_1-1, k_2-1, and one critical point. While the first portion of the theorem may be expected, the second portion shows a surprising contrast with *Walsh's Theorem* for polynomials. For, here the number of components in B_1 and B_2, rather than the total mass on B_1 and B_2, determines the number of critical points in C_1, C_2, and C_3. The latter result, however, follows from a study of the variation in direction of the force on each component of

B_1 and B_2 and the use of the Principle of Argument. Some of the above theorems have a counterpart in the case of Green's functions, with pole at a finite point for a doubly-connected region R containing the point, the counterpart being obtained by a mapping which sends this point to infinity.

Chapter VIII, entitled *Harmonic functions*, falls into two parts.

The first part considers a function $u(x, y)$ which is harmonic in a region R bounded by two sets of mutually disjoint Jordan configurations C and D, is continuous on $R+C+D$, is zero on C, and is unity on D. By definition, $u(x, y)=\omega(z, D, R)$, the harmonic measure of D with respect to R. At any point (x_0, y_0) in R, $u(x, y)$ may be written in the form $u(x_0, y_0)=\int_C \log r\, d\sigma-\int_D \log r d\sigma$, where $d\sigma>0$ and $\int_C d\sigma=\int_D d\sigma$. Now, introducing $v(x, y)$, the harmonic conjugate of $u(x, y)$, defining $f(z)=u(x, y)+iv(x, y)$, and differentiating, one obtains the representation

$$f'(z) = \int_C (z - t)^{-1} d\sigma - \int_D (z - t)^{-1} d\sigma.$$

Hence, the critical points of $u(x, y)$ are the points of equilibrium in the force field due to a spread of positive matter on C and of negative matter on D, with the total mass zero. Therefore, a Bôcher-type theorem is valid: if the disjoint circular regions M and N contain respectively C and D, then they also contain all the critical points of $u(x, y)$ in R with $m-1$ critical points in M and $n-1$ in N where m and n are the number of components of C and D respectively. Equally valid is also a result similar to Walsh's Cross-Ratio Theorem. Furthermore, if C is assumed to consist of a single Jordan curve J_0 and if J denotes the region bounded by J_0 and containing R, the region R may be provided with an NE geometry by mapping J upon the interior of the unit circle. Hence, the smallest convex NE polygon in J containing D must also contain all the critical points of $\omega(z, D, R)$. Finally, these theorems involving the harmonic measures of the closed components of D may be generalized to cover the harmonic measures of a finite number of arcs on D.

The second part of Chapter VII is concerned with linear combinations of Green's functions. In a region R, bounded by a Jordan curve C, there are given the distinct points $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n$. The linear combination

$$w(z) = \sum_1^m \lambda_k G(z, \alpha_k) - \sum_1^n \mu_k G(z, \beta_k)$$

is formed with $\lambda_j \geqq 0$ and $\mu_k \geqq 0$ for all j and k with $G(z, \gamma)$ denoting the Green's function for R with pole at γ. By applying earlier theorems with D as the locus $w(z) = M$, M sufficiently large, and with $u(z) = w(z)/M$, one may deduce, for the critical points of $w(z)$, a Lucas-type theorem in the special case $\mu_k = 0$ for all k and a Bôcher-type theorem in the general case. Furthermore, these theorems may be extended to regions of higher connectivity.

Chapter IX, entitled *Further harmonic functions*, is concerned with more general harmonic measures, linear combinations of harmonic measures and Green's functions, superharmonic functions, and brief applications of some other methods for investigating critical points.

Chapter IX begins by generalizing the previous integral representations. If $u(x, y)$ denotes a function, harmonic in a region R bounded by a finite Jordan configuration B and continuous on $R+B$, and if $f(z) = u(x, y) + iv(x, y)$ where $v(x, y)$ is the harmonic conjugate of $u(x, y)$, then

$$f'(z) = -(2\pi)^{-1}\int_B (z-t)^{-1}dv - i(2\pi)^{-1}[u(z-t)^{-1}]_B + i(2\pi)^{-1}\int_B (z-t)^{-1}du.$$

Thus, the critical points of $u(x, y)$ in this general case are the points of equilibrium in a force field due not only to a spread $-dv/2\pi$ of ordinary matter on B, but also to a spread $du/2\pi$ of skew matter on B and to skew particles corresponding to the middle term in the above formula. Without first mapping R upon the unit circle as was necessary in Chapter VIII, one may now obtain directly the following theorem. If $B = J + C$ where J is a Jordan curve and C a Jordan configuration disjoint from J, if A is an arc of J and if K is a circle which separates the interior points of A from points on C but not on K and from the interior points of $J - A$, then no critical point of $\omega(z, A, R)$ lies in R on K. This theorem may be extended to the harmonic measure of several arcs of Jordan configurations and, in greater generality, to linear combinations including either or both the harmonic measures of the separate Jordan arcs or configurations and the Green's functions for R with poles at several given points in R.

In the second part of Chapter IX, methods other than that of force fields are studied briefly. For example, the method of symmetry alone leads to regions free of the critical points in the case of functions which assume at a given point set greater values than at the corresponding reflected points. As another example topological methods may be used to prove that, if $u(x, y)$ is harmonic a region R bounded by a Jordan curve J, and continuous on $R+J$, and if $u(x, y) \geqq 0$ on an arc C of J and $u(x, y) \leqq 0$ and $J-C$, then no critical point of $u(x, y)$ lies in R on the locus $u(x, y)=0$.

The final part of Chapter IX deals with the potential of single and double layer distributions. The most striking result concerns a function $u(x, y)$ which is superharmonic in a region R interior to a Jordan curve C and harmonic in a subregion R' of R bounded by C and by a closed set B in R. If, in addition, $u(x, y)$ is continuous near C in $R+C$ and zero on C, then all the critical points of $u(x, y)$ in R' lie in the smallest NE convex set of R containing B. This theorem is a consequence of F. Riesz' integral representation for a function superharmonic in the interior of a unit circle. For, this representation permits $u(x, y)$ to be regarded as the potential of a certain spread of matter on B.

Thus, over the 376 pages of his text, the author adheres, with a remarkable, nearly perfect, consistency, not only to the single objective of studying the location of the critical points of various functions, but also to the single method of regarding these critical points as equilibrium points in fields of force due to suitable distributions of matter. He not only treats each individual topic with a local thoroughness but also indicates its relation to the preceding and succeeding topics. It is obvious that the manuscript and proofs were prepared and examined with unusual care, for the book seems to be not only free of mathematical errors, but also almost entirely of typographical errors. In short, this is a well-organized, thorough treatment of the subject. In the opinion of the reviewer, it is destined to serve, for a long time to come, as the principal reference book on the location of critical points.

Only in a minor way does Walsh's book overlap the reviewer's recent book in the Mathematical Surveys Series, entitled *The geometry of the zeros of a polynomial in a complex variable.* Specifically, the overlapping occurs between parts of Walsh's Chapters I and IV and of the reviewer's Chapters I, II, III and V. Such overlapping is of course unavoidable in any book aiming to be self-contained and in Walsh's book is not objectionable since Walsh usually treats the duplicated material by methods different from those of the reviewer.

With an exposition that is clear, complete, and well-illustrated with drawings and examples, the book may be recommended even to the general reader in analysis, geometry, or applied mathematics. Indeed, this reader will find in Walsh's book a refreshing change from the extreme abstractness of some present-day mathematics and perhaps he too will find it encouraging that so much new and important mathematics can still be discovered by relatively elementary methods.

MORRIS MARDEN

Anwendung der elliptischen Funktionen in Physik und Technik. By F. Oberhettinger and W. Magnus. (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, vol. 55.) Berlin, Springer, 1949. 8+126 pp.

The authors' collection, *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik*, which appeared some years ago, is now supplemented by a treatment of those applications of elliptic functions and integrals which arise in the study of a wide variety of physical and engineering problems. With the exception of some conformal maps, none of the results of the theory of elliptic functions are proved. All formulas used in the applications are, however, collected in the first chapter and, wherever desirable, have been supplemented by useful comments.

In the first chapter the authors study elliptic integrals of the first and second kind in Legendre's normal form, as well as the corresponding complete integrals; the elliptic integral of the third kind is not considered. A variety of expansions and transformation formulas are listed, together with a large number of integrals reducible to them. This is followed by the four theta-functions and their properties and by a similar treatment of the Jacobian elliptic functions. A somewhat briefer treatment is accorded the Weierstrass theory. The second chapter deals with the conformal mapping of ellipses and certain types of polygons. The third chapter is devoted to a large number of examples of electrostatic distributions in two dimensions which may be treated by means of elliptic functions. The fourth chapter deals with similar applications to problems in fluid dynamics. In particular, there are some problems on wind tunnels, such as the airfoil in a wind tunnel of elliptic cross-section. The fifth chapter is a collection of various unrelated physical problems, such as the pendulum and the potential due to a charged ellipsoid. Finally, the authors consider a problem of Chebyshev approximation which leads to elliptic functions.

A short, but useful, bibliography follows each chapter and the book concludes with a short set of tables of the Legendre integrals of the first and second kind.