THE NOVEMBER MEETING IN EVANSTON

The four hundred sixty-second meeting of the American Mathematical Society was held at Northwestern University on Friday and Saturday, November 24 and 25, 1950. In view of the fact that Northwestern University is celebrating its Centennial in 1951, this meeting of the Society was considered a portion of the centennial celebration. The attendance was about 160, including the following 132 members of the Society.

E. J. Akutowicz, L. U. Albers, A. A. Albert, W. R. Allen, J. J. Andrews, D. G. Austin, E. W. Banhagel, R. H. Bardell, D. Y. Barrer, R. G. Bartle, L. D. Berkovitz, H. R. Beveridge, R. H. Bing, L. M. Blumenthal, M. L. Boas, R. P. Boas, D. G. Bourgin, A. J. Brandt, Leonard Bristow, R. H. Bruck, R. C. Buck, E. L. Buell, J. W. Butler, Lamberto Cesari, Herman Chernoff, E. W. Chittenden, H. M. Clark, E. H. Clarke, M. D. Clement, E. G. H. Comfort, M. L. Curtis, Allen Devinatz, Flora Dinkines, N. J. Divinsky, W. F. Eberlein, M. H. M. Esser, G. W. Evans, W. A. Ferguson, W. H. Fleming, L. R. Ford, Evelyn Frank, J. C. Freeman, R. E. Fullerton, M. P. Gaffney, Cornelius Gouwens, L. M. Graves, Harold Greenspan, L. W. Griffiths, M. M. Gutterman, Franklin Haimo, P. R. Halmos, Melvin Henriksen, Fritz Herzog, D. L. Holl, T. C. Holyoke, A. S. Householder, C. C. Hsiung, H. K. Hughes, Meyer Jerison, L. W. Johnson, L. H. Kanter, Irving Kaplansky, William Karush, L. M. Kelly, D. E. Kibbey, E. H. Kingsley, Erwin Kleinfeld, Marc Krasner, M. Z. Krzywoblocki, H. G. Landau, Walter Leighton, D. J. Lewis, G. R. Livesay, A. P. Loomer, R. E. Lowney, Saunders MacLane, H. M. MacNeille, D. M. Merriell, J. M. Mitchell, C. W. Moran, E. J. Moulton, H. T. Muhly, S. B. Myers, E. A. Nordhaus, R. Z. Norman, T. E. Oberbeck, E. N. Oberg, Rufus Oldenburger, H. W. Oliver, E. H. Ostrow, O. G. Owens, Gordon Pall, George Piranian, D. H. Potts, G. B. Price, A. L. Putnam, W. T. Reid, Haim Reingold, Maxwell Rosenlicht, A. E. Ross, E. H. Rothe, J. C. Rothe, R. G. Sanger, R. H. Scherer, O. F. G. Schilling, J. E. Schubert, W. T. Scott, I. E. Segal, Esther Seiden, D. H. Shaftman, H. A. Simmons, M. L. Slater, D. M. Smiley, M. F. Smiley, E. H. Spanier, R. H. Stark, R. L. Sternberg, T. T. Tanimoto, H. P. Thielman, N. S. Thompson, E. F. Trombley, Bryant Tuckerman, W. R. Utz, Bernard Vinograde, D. R. Waterman, L. R. Wilcox, R. S. Wolfe, F. M. Wright, L. C. Young, J. W. T. Youngs, Daniel Zelinsky, J. L. Zemmer.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor D. G. Bourgin, of the University of Illinois, addressed the Society on *Classes of transformations and bordering transformations* at 2:15 p.m. Friday, November 24. Presiding officer at this lecture was Dr. H. M. MacNeille, Executive Director of the Society.

There were, in addition to the lecture by Professor Bourgin, four sessions for the presentation of contributed papers, two on Friday and two on Saturday. Presiding officers at these sessions were Professors W. T. Reid, L. M. Graves, W. T. Scott, and R. H. Bruck.

There was a tea for visiting members and their guests in the Faculty Lounge of the Technological Institute Building on Friday afternoon. For the first time in many years, a dinner was held in connection with a meeting of the Society in the Chicago area. This was a very pleasant affair in which Dean Tebbutt welcomed the Society on behalf of the University and your Associate Secretary replied. The Executive Director, Dr. H. M. MacNeille, was then called upon by the toastmaster, Professor L. R. Ford, to make a few remarks. He stated briefly some of the tasks of the New York office and concluded with a few comments in the nature of a report from foreign visitors to the recent International Congress of Mathematicians.

Abstracts of papers presented follow below. Abstracts followed by the letter "*t*" were read by title. Abstracts 446 and 447 in the September issue of this Bulletin and abstracts 453, 457–460 in the November issue were also read by title at this meeting. Paper number 27 was read by Professor Fullerton, 32 by Professor Herzog, 35 by Professor Piranian, and 52 by Professor Young. Mr. Nemmers was introduced by Dr. H. M. MacNeille and Mr. Chu was introduced by Professor Bernard Vinograde.

Algebra and Theory of Numbers

14*t*. R. H. Bruck: *The definitive associativity theorem for alternative rings.*

Let R be an alternative ring with associator $(x, y, z)=xy\cdot z-x\cdot yz$. A subset A of R (not necessarily closed under subtraction or multiplication) is called associative if $(A, A, A)=0$. The main theorem states: Every maximal associative subset of an alternative ring R is an associative subring. The proof holds if R is commutative or if $3(x, y, z)=0$ implies $(x, y, z)=0$, but these restrictions are probably irrelevant. Assuming the well-ordering postulate, the theorem contains as special cases the associativity theorems of Artin and Moufang, and the proof (aside from well-ordering) is exceedingly simple. A like theorem holds for commutative Moufang loops and presumably for noncommutative Moufang loops. (Received May 17, 1950.)

15*t*. Harvey Cohn: *On stable maxima for the product of homogeneous forms.* I.

For every real matrix a_{ij}, det $a_{ij}>0$, we define $\Phi(a)=\min |a\times p|$ where $a\times p =(\prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij}p_j)/\det a_{ij}$ and p_j varies over all integer n-tuples, not all zero. Let $|a-a^*|=(\sum_{ijk} (a^*_{ij}a_{ik}-a^*_{ik}a_{ij})^2)^{1/2}$ be a measure of the difference between the matrices, vanishing when the rows are proportional in order. Henceforth the author calls the value $\Phi(a^*)$ a (relative) *stable maximum* if positive numbers ϵ, ρ and a set of q $(=n(n-1)+1)$ integral n-tuples $p^{(k)}$ exist such that $|a^*\times p^{(k)}|=\Phi(a^*)$ and, for $|a-a^*|<\epsilon$, and $k=k(a)$, this inequality holds: $|a\times p^{(k)}|<\Phi(a^*)-\rho|a-a^*|$. Referring to quadratic forms $a\times p=F(p_1, p_2)/(\text{disc } F)^{1/2}$, with coefficients proportional to integers, the author shows that $\Phi(a)$ provides a stable maximum if and only if the pair of diophantine equations $F(p_1, p_2) = \pm \min |F(p, q)|$ are both solvable in integers. (Received October 6, 1950.)

16t. Harvey Cohn: *On the finite determination of critical lattices for a convex body.*

The problem can be restricted without loss to lattices in d-space which have a set of d independent vectors α_i on the surface of the convex body. Considering first lattices where such a set is a basis, the author shows that the conditions that a convex body contains no lattice point in its interior except the origin can be reduced from the infinite set $\Phi(\sum_{i=1}^{d} p_i\alpha_i) \geqq 1$ where the integers p_i run over all n-tuples except the origin, to the finite set where $(\sum p_i^2)^{1/2} \leqq C_d d^{d/2}$ and where C_d is the constant defined in the author's previous abstract, *On finiteness conditions for a convex body in hyperspace* (Bull. Amer. Math. Soc. Abstract 56-3-104). In the event that the α_i do not form a basis, they are still shown to be of bounded index for each dimension, and a slight modification of the argument suffices. (Received October 6, 1950.)

17t. R. P. Dilworth and J. E. McLaughlin: *Distributivity.*

Let ϕ be an imbedding operator over a lattice L. L is said to be ϕ-distributive if $x \cap \phi(S) = \phi(x \cap S)$ for all elements x and all subsets of S of L. It is shown that L is ϕ-distributive if and only if L_ϕ is infinitely (meet) distributive. It is also shown that every Boolean algebra is ν-distributive where ν is the normal imbedding operator. Two important theorems on Boolean algebras—the Tarski-von Neumann Theorem and the Stone-Glivenko Theorem are consequences of these theorems. (Received October 5, 1950.)

18. Franklin Haimo: *Groups with a certain condition on conjugates.* Preliminary report.

Let G be a nilpotent group, and let M, a positive integer, be a uniform bound on the number of conjugates that any element of G may have. Then there are "large" integers n for which the operation $x \rightarrow x^n$ is a central endomorphism of G. (Received October 11, 1950.)

19t. H. T. Muhly: *Integral bases and varieties multiply of the first species.*

If V is an absolutely irreducible r-dimensional algebraic variety over a ground field k of characteristic zero, and if $\mathfrak{o} = k[x_0, x_1, \cdots, x_n]$ is the ring generated by the homogeneous coordinates of the general point of V, then a necessary and sufficient condition for the existence of an independent linear base for $\mathfrak{o}$ over the ring generated by a suitably chosen set of independent variables is that V be r times of the first species in the sense of Dubreil (*Sur la dimension des ideaux des polynomes*, Journal de Mathématiques (9) vol. 15 (1936)). It is shown that if $\mathfrak{a}$ is the ideal of V in the ring $R = k[X_0, X_1, \cdots, X_n]$ of $n+1$ indeterminates over k, and if $\mathfrak{a}_0 = \mathfrak{a}$, $\mathfrak{a}_i = \mathfrak{a}_0 + RX_1 + RX_2 + \cdots + RX_i$, $i = 1, 2, \cdots, r$, $\mathfrak{a}_{r+1} = \mathfrak{a}_0 + RX_0 + RX_1 + \cdots + RX_r$, then a necessary and sufficient condition for the $\mathfrak{a}$ residues $x_0, x_1, \cdots, x_r$ of $X_0, X_1, \cdots, X_r$ to be an admissible set of independent variables for $\mathfrak{o} \cong R/\mathfrak{a}$, and for $\mathfrak{o}$ to possess an independent linear base over $k[x_0, x_1, \cdots, x_r]$, is that the characteristic function $\chi(\mathfrak{a}, h)$ of $\mathfrak{a}$ satisfy the difference equations, $\Delta^j\chi(\mathfrak{a}, h) = \chi(\mathfrak{a}_j, h)$, $j = 1, 2, \cdots, r+1$, for all values of h, where Δ denotes the unit difference operator. (Received August 21, 1950.)

20t. F. E. Nemmers: *A theorem for alternative rings.*

Let a, b, c be elements of an alternative ring. The author proves the identity $(a, (a, b, c), c) = (a, b, c)(ca-ac) = (ac-ca)(a, b, c)$ and uses this identity to prove the theorem "An alternative division ring R satisfying $(a, (a, b, c), c) = 0$ for every $a, b, c \in R$, is a field." A corollary to this theorem is "In an alternative ring in which $6a = 0$ implies $a = 0$, the unit cannot be expressed as an associator." (Received October 9, 1950.)

21. G. B. Price: *Bounds for determinants with dominant principal diagonal.*

Let (a_{ij}) be an $n \times n$ matrix of complex numbers a_{ij}, and let D be its determinant. Assume that for each row the absolute value of the element in the principal diagonal is greater than the sum of the absolute values of all other elements in the same row (for references to the literature concerning these determinants see a paper by Olga Taussky in Amer. Math. Monthly vol. 56 (1949) pp. 672–676). Set $m_i = |a_{ii}| - \sum_{j=i+1}^{n} |a_{ij}|$ and $M_i = |a_{ii}| + \sum_{j=i-1}^{n} |a_{ij}|$ for $i = 1, 2, \cdots, n$. It is proved in this note that $0 < m_1 m_2 \cdots m_n \leqq |D| \leqq M_1 M_2 \cdots M_n$, and several generalizations and extensions are given. The lower bound for $|D|$ stated in the last sentence is an improvement of a similar lower bound given by Ostrowski (Bull. Sci. Math. (2) vol. 61 (1937) pp. 19–32). The proofs depend on a lemma which gives bounds for the solution of a certain system of linear equations. (Received October 2, 1950.)

22t. W. R. Utz: *Powers of a matrix over a lattice.*

In this note powers, under ordinary matrix multiplication, of a matrix with elements in a lattice are investigated. Among other results is a generalization of a theorem due to Sanders (*A linear transformation whose variables and coefficients are sets of points*, Bull. Amer. Math. Soc. vol. 48 (1942) p. 442) to matrices with elements in a distributive lattice. (Received October 2, 1950.)

23t. N. A. Wiegmann: *A polar form for a matrix in a field not of characteristic two.*

It is known that every matrix A in the complex field has a polar representation and that if A is normal, the polar matrices commute. A matrix A with elements in a field $K \supseteq F$ not of characteristic two is considered and a generalization of this polar form is obtained. It is shown that when U is an involution over F and A is defined to be normal if $AA^U = A^U A$, then A is normal if and only if $A^U = VA = AV$ where V is a U-orthogonal matrix. This is seen to include the complex case as a special instance. Possibilities for the modification of certain restrictions are considered as are special cases of this type of normal matrix. (Received October 4, 1950.)

Analysis

24. L. U. Albers: *Use of Leray-Schauder methods on the existence and boundedness of solutions of quasi-linear partial differential equations of parabolic type.*

The quasi-linear partial differential equation of parabolic type is studied. The space variables range over the interior of a simply connected region in Euclidean n-space and the boundary of this region must have certain continuity properties. The time variable ranges over the interval $(0, T)$ where the size of T depends on the properties of the source function. Moreover the space region mentioned above is allowed to vary continuously with time in the interval $(0, T)$ in certain restricted ways. The partial differential equation problem is transformed into a mapping problem in Banach spaces of the type studied by Leray and Schauder, and the existence problem is solved at that level. The implications of the basic results for the conditions on boundary values and properties of the source function are studied. (Received October 10, 1950.)

25. R. P. Boas: *Completeness of cosines with phase shifts.*

It has been shown by V. A. Ditkin (Uspehi Matematičeskih Nauk N.S. vol. 5 (1950) pp. 196–197) that the set $\{\cos(nx+q_n)\}_{n=0}^{\infty}$ is complete on $(0, \pi)$ with respect to the class of integrable functions provided that $0 \leqq q_n < \pi/2$ for $n=0, 1, 2, \cdots$. The author extends this result, using complex-variable methods (different from the method used by Ditkin), to sets of the form $\{\cos(\lambda_n x+q_n)\}_{n=0}^{\infty}$. For example, if $|n-\lambda_n| \leqq 2\delta/\pi < 1/2$, $\delta > 0$, this set is complete if $\delta \leqq q_n < (\pi/2)-\delta$. (Received October 10, 1950.)

26. Lamberto Cesari: *On the absolute extremals for the integrals on parametric surfaces.*

The following existence theorem is obtained: *Each positive definite semiregular integral $J(S)$ has absolute minimum in the class of all surfaces of finite Lebesgue area, contained in a closed bounded convex part A of the (xyz)-space, and whose boundary curve is a given closed Jordan curve C provided that C is boundary curve of at least one surface of finite Lebesgue area in A.* This theorem, in which the condition of the boundedness of A can be removed, extends to general parametric surfaces a theorem of Tonelli for curves. The same theorem extends an analogous statement due to McShane for integrals $J(S)$ in which the function involved does not depend upon the point in the space. Another particular case is the problem of Plateau. The proof of the above existence theorem is based only on the concept of lower semicontinuity, on geometrical considerations which arise in the theory for Lebesgue area, on the Eilenberg inequality applied to polyhedral surfaces, and on a detailed study of the known smoothing processes and especially of a new one for polyhedral surfaces. Analogous results have been communicated recently by J. M. Danskin and A. G. Sigalov as obtained by different methods. (Received November 25, 1950.)

27. Lamberto Cesari and R. E. Fullerton: *On regular representations of surfaces.*

Let S be a Frechet surface. A representation Φ: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$; $0 \leqq u, v \leqq 1$, of S is said to be regular if there exists a countable set of horizontal line segments $\{\eta_i\}$, $0 \leqq u \leqq 1$, $v=k_i$, and a countable set of vertical segments $\{\xi_j\}$, $0 \leqq v \leqq 1$, $u=c_j$, each dense in the unit square and such that the projections of the image under Φ of each segment on the coordinate planes are of Lebesgue two-dimensional measure zero. It is shown that any nondegenerate surface of finite Lebesgue area possesses a regular representation. The methods used are geometric and make no use of the Dirichlet integral. (Received October 11, 1950.)

28. Herman Chernoff: *An extension of a result of Liapounoff on the range of a vector measure.*

Liapounoff proved that if $\mu_1, \mu_2, \cdots, \mu_n$ are countably additive finite measures, the range of values of the vector $\mu(E)=(\mu_1(E), \mu_2(E), \cdots, \mu_n(E))$ is a bounded closed set and in the nonatomic case convex. If μ_{it}, $i=1, 2, \cdots, k$: $t=1, 2, \cdots, n_i$, are countably additive finite measures and $\{(E_1, E_2, \cdots, E_k)\}$ is the totality of decompositions of a space X into k pairwise disjunct measurable sets, then the range of $\psi=(\mu_{11}(E_1), \mu_{12}(E_1), \cdots \mu_{1n_1}(E_1), \mu_{21}(E_2), \cdots, \mu_{kn_k}(E_k))$ is bounded and closed and in the nonatomic case, convex. (Received October 18, 1950.)

29. J. T. Chu: *Generalized hermitian operators in Hilbert space. Preliminary report.*

Let X be an operator in Hilbert space S characterized by: (1) X^* exists, (2) $D(X) \cap D(X^*)$ generates S, and (3) $X=X^*$ over $D(X)\cap D(X^*)$. X is a generalization of a Hermitian operator. If linearity and boundedness are also assumed, then X is Hermitian. So to get nontrivial results, we must assume that X is unbounded. In some respects X is like a Hermitian operator. In fact it is a nonsymmetric extension of a Hermitian operator. We prove some sets of necessary and sufficient conditions that a nonsymmetric extension of a Hermitian operator be an operator of this kind. A more concrete treatment is obtained by using infinite matrices. However here we must assume that X is closed and linear. Then X is represented by matrices which are "almost" Hermitian. And conversely, unitary transformations of Hermitian matrices under certain conditions yield operators of this kind. (Received October 12, 1950.)

30. Allen Devinatz: *Integral representations of positive definite functions.*

Following the ideas introduced by N. Aronszajn (Proc. Combridge Philos. Soc. vol. 39 (1943) p. 133; Trans. Amer. Math. Soc. vol. 68 (1950)) the author investigates transformations in reproducing kernel spaces. Necessary and sufficient conditions are obtained for a function $f(x)$, $0\leqq x<\infty$, to be represented uniquely as the integral $\int_0^\infty t^x dV(t)$, where $V(t)$ is a bounded monotone increasing function. For $-\infty<x<\infty$, necessary and sufficient conditions are given for $f(x)$ to be represented by the integral $\int_\epsilon^\infty t^x dV(t)$, where $V(t)$ is as before and $\epsilon>0$. In this case it is shown that the representation is always unique. Necessary and sufficient conditions are also obtained for uniquely representing a function $f(x, y)$ as the integral $\int_0^\infty\int_{-\infty}^\infty t_1^x e^{it_2 y} dV(t_1, t_2)$. The main condition on these functions is that $f(x_1+x_2)$ or $f(x_1+x_2, y_1-y_2)$ are positive definite functions. The main idea of the proof is to set up Abelian systems of self-adjoint or normal transformations and use these systems in much the same manner as groups of unitary transformations are used to prove the Bochner theorem from Stone's theorem. Using discrete sequences as the domains of these functions, various types of moment problems may be solved. Extensions to higher-dimensional cases present no difficulties. (Received October 12, 1950.)

31. M. P. Gaffney: *The harmonic operator for exterior differential forms.*

For Riemannian manifolds, de Rham and Kodaira have defined the generalized harmonic operator $\Delta(=d\delta+\delta d)$. If its domain is properly chosen, Δ can be viewed as a Hilbert space transformation. Following a suggestion of M. H. Stone's, the author has used the Friedrichs mollifier to develop the properties of Δ from this point of view (thus avoiding the use of integral equation theory). To obtain results in the noncompact case the concept of "negligible boundary" is introduced: if α^{p-1}, $d\alpha$, β^p, and

$\delta\beta$ are all square integrable then $(d\alpha, \beta) = (\alpha, \delta\beta)$. Sufficient conditions are given for negligible boundary. The closure of Δ (with suitable domain) is proved self-adjoint (a necessary prelude to the use of the spectral theorem). All characteristic forms of Δ are proved to be of class C^2. In the compact case a direct proof is given that the inverse of Δ is completely continuous. (Received October 12, 1950.)

32. Fritz Herzog and George Piranian: *On the univalence of functions whose derivatives have a real positive part.*

A domain D is said to have property (U) provided each function whose derivative has a positive real part throughout D is schlicht in D. K. Noshiro showed that every convex domain has property (U) (Journal of the Faculty of Science, Hokkaido Imperial University, Series I, vol. 2 (1934–1935) p. 151). A domain is defined to be *almost convex* provided each pair of circles in D can be connected by a straight line segment in D. The *set of deficiency* of a domain is its complement relative to the interior of its convex hull. The authors prove the following: For a domain D to have property (U), it is sufficient that D be almost convex and necessary that the set of deficiency of D be totally disconnected. (Received August 23, 1950.)

33. A. S. Householder: *Polynomial iterations to roots of algebraic equations.*

By an iteration to a root ξ of an equation $f(x) = 0$ is meant a rule $\phi(x)$ such that for any x_0 in some neighborhood of ξ the sequence defined by the recursion $x_{i+1} = \phi(x_i)$ converges to ξ. The iteration is of order r in case $\phi^{(r)}$ is the derivative of lowest order which does not vanish at ξ. It is known that when f is analytic, an iteration ϕ of any specified order r can be constructed, and Domb, Proc. Cambridge Philos. Soc. vol. 45 (1949) pp. 237–240, has shown that when f is a polynomial one can make ϕ also a polynomial. The purpose of this note is to establish a simple algorithm for this. If d is the highest common divisor of f and f', $g = f/d$, and p and q are any polynomials satisfying $pg' + qg = 1$, then $\phi_1 = x - gp$ is a polynomial iteration of second order, and $\phi_r = \phi_{r-1} + (-)^r p_r g^r / r$ is a polynomial iteration of order $r+1$ provided $p_1 = p$ and $p_r = pp'_{r-1} - (r-1)qp_{r-1}$. (Received October 9, 1950.)

34. L. H. Kanter: *On the zeros of the parametric derivatives of the ultraspherical polynomials.*

The notation used is that of G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23. Let $\partial P_n(x, \lambda)/\partial\lambda = G_n(x, \lambda)$. Since $P_n(x_\nu, \lambda) \equiv 0$ in λ, it follows that $[\partial P_n(x, \lambda)/\partial\lambda + (\partial P_n/\partial x)x'(\lambda)]_{x=x_\nu} = 0$. From the fact that $R_n(x, \lambda)/P_n(x, \lambda)$ has at most one zero between two successive zeros of $P_n(x, \lambda)$ of the same sign, and the relation $R_n(x_\nu, \lambda) + P_n'(x_\nu, \lambda)x_\nu'(\lambda) = 0$, the following theorem is deduced: the positive zeros of $R_n(x, \lambda)$ interlace with the positive zeros of $P_n(x, \lambda)$; a similar theorem holds for the negative zeros. Also, like theorems hold for the polynomials $G_{n-2}(x, \lambda) = \partial(k_n^{-1} P_n)/\partial\lambda$ and for $F_n(x, \lambda) = R_n(x, \lambda) + cP_n(x, \lambda)$, where c is an arbitrary constant. (Received October 9, 1950.)

35. A. J. Lohwater and George Piranian: *Note on certain sets of harmonic measure zero.* Preliminary report.

This paper continues the search for conditions on a set E on the boundary of a Jordan region R that are necessary or sufficient in order that, for the conformal mappings f of R upon the unit disc, $f(E)$ should be a set of one-dimensional Lebesgue measure zero. It is known that the two-dimensional measure of a Jordan curve need not be zero [Osgood (Trans. Amer. Math. Soc. vol. 4 (1903) pp. 107–112), R. L. Moore and J. R. Kline (Ann. of Math. (2) vol. 20 (1918) pp. 218–223)]. The authors construct a Jordan curve that has a subset whose two-dimensional measure is positive, and whose harmonic measure is zero. (Received October 2, 1950.)

36. R. E. Lowney: *A boundary value problem involving an exponential turning point.*

The differential system $u''(x)+\rho^2q(x)u(x)=0$; $u(-a)=u(b)=0$; a, $b>0$; has been studied by R. E. Langer and J. Barron wherein $q(x)$ vanishes as a power of x. In the present paper, $q(x)$ is taken as $e^{-2/|x|}/x^4$, $x\neq 0$; $q(0)=0$, and $a=b$. The removeable discontinuity in $q(x)$ is reflected in the second independent solution of the differential equation. The eigenvalues are found to consist of two real infinite sets asymptotically approaching a regularly spaced set, one from above and the other from below. The expansion of a function in terms of eigenfunctions is discussed, and conditions for convergence determined. It is found that conditions for convergence at the point $x=0$ are slightly more restrictive than those for convergence at any other point of the interval. (Received October 9, 1950.)

37. O. G. Owens: *Homogeneous Dirichlet problem for inhomogeneous ultrahyperbolic equation.*

Let λ_n $(n=1, 2, 3, \cdots)$ denote the characteristic values associated with the wave-equation, $\phi x_1x_1+\phi x_2x_2+\lambda\phi=0$, and with the region $G(X)$ of cartesian (x_1, x_2)-space. Similarly, let μ_m $(m=1, 2, 3, \cdots)$ denote the characteristic values associated with the region $G(Y)$ of cartesian (y_1, y_2)-space. Then, the solution of the Dirichlet problem for the cross product region $G=G(X)\times G(Y)$ and the ultrahyperbolic equation $^ux_1x_1+{}^ux_2x_2-{}^uy_1y_1-{}^uy_2y_2=f(x_1, x_2, y_1, y_2)$ is unique, provided $|\lambda_n-\mu_m|$ is never zero. Furthermore, if f vanishes on the boundary of G and if $|\lambda_n-\mu_m|$ has a positive lower bound, then there is a unique solution of the ultrahyperbolic equation vanishing on the boundary of G. An example shows that there are domains $G=G(X)\times G(Y)$ for which $|\lambda_n-\mu_m|$ is bounded away from zero. (Received October 11, 1950.)

38t. A. R. Schweitzer: *Functional equations associated with Grassmann's extensive algebra.*

Two classes of functional equations are discussed. I. Equations in iterative compositions of functions (including the author's quasi-transitive functional equations) generalizing Grassmann's relation $a-b=(a-r)-(b-r)$, *Gesammelte Werke*, vol. 1, part 1, p. 373. II. Equations connected with Grassmann's inner product; ibid. pp. 11, 345, 349. Let $E_1, E_2, \cdots, E_n$ $(n=2, 3, \cdots)$ be independent vectors in Grassmann space; let $X_i=x_{i1}E_1+x_{i2}E_2+\cdots+x_{in}E_n$ $(i=1, 2, 3)$ and assume $X_i^2=f(x_{i1}, \cdots, x_{in})$. Then the identity $(X_1+X_2+X_3)^2=X_1^2+X_2^2+X_3^2+2X_1X_2+2X_2X_3+2X_3X_1$, where $2X_iX_j=(X_i+X_j)^2-X_i^2-X_j^2$, leads to the relation $(X_1+X_2+X_3)^2-(X_1+X_2)^2-(X_2+X_3)^2-(X_3+X_1)^2+X_1^2+X_2^2+X_3^2=0$. Replacing the indicated inner squares by corresponding functional expressions, the author obtains a functional equation which is interpreted as a special case of a functional equation due to M. Fréchet, Nouv. Ann. Math. (4) vol. 9 (1909) pp. 145–162. Reference is made to G. van der Lijn, Bull. Sci. Math. vol. 64 (1940), part 1, pp. 55–80, 102–112, 163–196. (Received October 9, 1950.)

39. E. F. Trombley: *Some boundary value problems involving functions of two complex variables.*

If a function of two complex variables $f(z_1, z_2)$ is (i) regular when the imaginary parts of z_1 and z_2 are both greater than zero (call this region E) and (ii) $\sup_{y_1,y_2>0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x_1+iy_1, x_2+iy_2)|^p dx_1 dx_2 = M^p < \infty$ where $p>0$ and M is a constant, then the iterated limit $\lim_{y_1\to 0} \lim_{y_2\to 0} f(x_1+iy_1, x_2+iy_2)$ $(=F(x_1, x_2))$ exists almost everywhere and is almost everywhere equal to the limit of $f(w_1, w_2)$ as (w_1, w_2) approaches (x_1, x_2) along any path nontangential to the distinguished boundary of E. Further $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(x_1, x_2)|^p dx_1 dx_2 = M^p$. If $p \geqq 1$, then $f(z_1, z_2)$ is represented by both the Cauchy and Poisson integral of $F(x_1, x_2)$. In this case $\sup_{y_1>0} \int_{-\infty}^{\infty} |f(x_1+iy_1, z_2)|^p dx_1 < \infty$ for almost every z_2. (Received October 12, 1950.)

40. D. R. Waterman: *On some high-indices theorems.*

The Hardy-Littlewood "high-indices" theorem asserts that for lacunary series Abel summability implies convergence. That is to say that if $f(s) = \sum_1^\infty a_n e^{-\lambda_n s}$ where $\lambda_{n+1}/\lambda_n \geqq q > 1$ for $n = 1, 2, \cdots$, and $\lim_{s\to +0} f(s)$ exists, then $\sum_1^\infty a_n$ converges. In his proof of this theorem Ingham demonstrates the basic result $|a_n| \leqq A_q \sup f(s)$ where A_q depends on q alone. Absolute Abel summability is defined by the condition that $f(s)$ be a function of bounded variation in $[0, \infty]$. Zygmund has shown that absolute Abel summability implies absolute convergence for lacunary series. Important there is the inequality $\sum_1^\infty |a_n| \leqq A_q \int_0^\infty |f'(s) ds$. Here the author shows that $\sum_1^\infty |a_n|^m \lambda_n^{m-1} \leqq A_{q,m} \int_0^\infty |f'(s)|^m ds$ where $A_{q,m}$ depends on q and m alone. The Hardy-Littlewood theorem and the Zygmund theorem are the cases corresponding to $m = \infty$ and $m = 1$ respectively. Also given are the results $\sum_1^\infty |a_n|^m \leqq A_{n,q,m} \int_0^\infty (1-e^{-s})^{m-1} |f'(s)|^m ds$ where $0 < h \leqq \lambda_1$, and $\sum_1^\infty |a_n|^m \lambda_n^{m-\beta} \leqq A_{n,q,m,\beta} \cdot \int_0^\infty (1-e^{-s})^{\beta-1} |f'(s)|^m ds$ for $\beta \leqq m$. (Received August 22, 1940.)

Applied Mathematics

41. E. J. Akutowicz. *On the so-called phase problem for Fourier integrals.*

In the present note the following theorem is proved: Let ρ be a given real or complex-valued function on $-\infty < x < +\infty$ belonging to $L^1 \cap L^2$ and vanishing for $x > x_0$. Let $\mathcal{C}$ be the class of all such functions, where x_0 may vary with the function. Let $\hat{\sigma}$ denote the Fourier transform of σ. Then the only functions $\sigma \in \mathcal{C}$ such that $|\hat{\sigma}| = |\hat{\rho}|$ are of the form $\exp (ic) \cdot \rho(x + \text{const.})$, where c is a real constant. This result is related to a theoretical question of some urgency in crystallography. Namely, to what extent do the absolute values of the Fourier coefficients of a positive periodic function determine that function? (Received September 25, 1950.)

42. M. Z. Krzywoblocki: *On the generalization of Friedrichs' theorem on the nonvanishing of the Jacobian in transonic flow.*

Nikolsky and Taganov proved the theorem on the non-occurrence of a limiting line at the boundary or in the interior of the supersonic irrotational flow region of an inviscid fluid. Friedrichs presented a more complete theorem in which he not only had shown these facts in a different way, but in addition, he proved that the limiting line cannot first appear on the curve of sonic flow of such a fluid. Friedrichs' proof shows precisely that the Jacobian cannot vanish somewhere in the flow domain. In the present paper the author extends Friedrichs' proof on the nonvanishing of the Jacobian to rotational and viscous fluid flows. The relation between Jacobian and the limiting line in such fluids is treated in another paper by the author. (Received September 25, 1950.)

43*t*. H. E. Salzer: *Formulas for finding the argument for which a function has a given slope.*

The problem of finding where a function $f(x)$ that is tabulated at equal intervals h has a given slope, that is, given $f'(x) \equiv f'(x_0+ph)$, to find p, arises very frequently in numerical computation, especially when locating a maximum or minimum point where $f'(x_0+ph)=0$. Reversion of the series in p which is obtained by differentiating any one of the direct interpolation formulas for $f(x_0+ph)$ is more convenient than (1) calculation of the derivatives at the tabular points followed by inverse interpolation or (2) subtabulation of the derivative around an estimated value of p. These formulas give the coefficients of the reverted series for p directly in terms of the tabular entries so that no differences are necessary. The formulas cover the cases where n-point Lagrangian interpolation formulas are needed in direct interpolation, for $n=3$, 4, 5, 6, and 7. (Received October 10, 1950.)

Geometry

44*t*. A. R. Schweitzer: *On the derivation of the regressive product in Grassmann's geometrical calculus.* II.

Regressive products are considered only in association with an n-simplex S_n ($n=2, 3, \cdots$) with vertices $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ derived numerically from the independent points $\epsilon_1, \epsilon_2, \cdots, \epsilon_{n+1}$. Then by $P(\alpha_1, \alpha_2, \cdots, \alpha_{n+1})$ the author denotes a regressive product and by $V(\alpha_1, \alpha_2, \cdots, \alpha_{n+1})$ its value; and $P(\alpha_1, \alpha_2, \cdots, \alpha_{n+1}) = V(\alpha_1, \alpha_2, \cdots, \alpha_{n+1})$ is assumed to be the complete statement governing P. The author discusses outer products whose factors are outer products in analogy with determinants having determinants as constituents (including reciprocal determinants). The special case $n=3$ is considered with $P_1=\alpha_1\alpha_2\alpha_3\cdot\alpha_1\alpha_2\alpha_4$ and $P_2=\alpha_1\alpha_2\alpha_3\cdot\alpha_1\alpha_2\alpha_4\cdot\alpha_1\alpha_3\alpha_4$. Let V_1 and V_2 be the values of P_1 and P_2 respectively. Extending a suggestion of Grassmann (*Gesammelte Werke*, vol. 1, part 1, p. 335) the author assumes $V_1=r_1\cdot\alpha_1\alpha_2$ and $V_2=r_2\cdot\alpha_1$ where r_1 and r_2 are numerical functions of the α's and the ϵ's which remain to be specialized. With the aid of elementary properties of determinants it is found that $r_1=c_1q$ and $r_2=c_2q^2$ where $q=[\alpha_1\alpha_2\alpha_3\alpha_4]/[\epsilon_1\epsilon_2\epsilon_3\epsilon_4]$ and c_1 and c_2 are undetermined numerical functions of the ϵ's. (Received October 9, 1950.)

45. Esther Seiden: *Some theorems in finite projective geometry.*

The purpose of this paper is to investigate the properties of a set consisting of a maximum number of no three collinear points in finite projective plane $PG(2,2^n)$—with three homogeneous coordinates, each taking on 2^n values. It is shown that $2^{n-1}+2$ points of the configuration determine uniquely the remaining 2^{n-1} points. The problem studied is: Given 4 no three collinear points to determine all the set of no three collinear points containing these four points. The number of such sets consisting of a conic and the point of intersection of the tangents to the conic is found to be 2^n+2. For $n=3$, the conics form the only sets of no three collinear points. However, for n greater or equal to 4 it is proved that there always exist sets of no three collinear points which do not form a conic. (Received October 11, 1950.)

Statistics and Probability

46*t*. J. J. Andrews: *On the admissibility of time series.*

In two papers of Regan (Trans. Amer. Math. Soc. vol. 36 (1934) pp. 511–529 and Amer. J. Math. vol. 43 (1936) pp. 867–873) it was shown that the laws pertaining to the distribution function of time series were consistent. In the first paper an ideal series having constant probability was constructed by the use of the admissible numbers of Copeland. In the problem of time series with variable probability Regan transforms the series obtained in his first paper. This paper combines the results of these two papers. An ideal time series is constructed so that the laws pertaining to the case of variable probability are satisfied and then with suitable restrictions on this series with its corresponding distribution function the case where the probability is constant is solved. The number of essential theorems used here to establish the same results can be reduced to two, with corollaries forming the other results. Regan considered intervals of the form 2^{-r+1} and for the probability of at least one point he also considered rational intervals not of the form 2^{-r+1} and irrational intervals. Besides these intervals, this time series is shown to be valid for a set of points E whose frontier points are of measure zero when applied to the probability of at least one point in the set E. (Received October 9, 1950.)

47*t*. J. H. Curtiss: *Sampling methods applied to differential and difference equations with special reference to equations of the elliptic type.*

The paper discusses at some length the relationship between random walks in bounded regions and elliptic differential and difference operators, stressing statistical consideration useful in applying the "Monte Carlo method" to the numerical solution of boundary value problems. Although the basic theorems are due to Courant-Friedrichs-Lewy and Petrowski, a number of results believed to be new are presented. These include a uniform bound for the mean duration of general random walks starting anywhere in the region; derivation of the nonhomogeneous difference equation satisfied by the mean duration of a walk on a rectangular lattice; various explicit formulas and special bounds for the mean duration; an estimate of the degree of approximation of the solution of the difference equation to that of the differential equation (the estimate is in terms of the mean duration of the walk); formulas for the dispersion of the random walk statistics (including the dispersion of the number of returns to a given point); a discussion of "importance sampling" and bookkeeping procedures. In connection with the last item a result due to Wasow is proved which shows that if a random walk starts at P and the particle visits a second point P' one or more times, then these visits can profitably be considered as providing a bonus of one, but only one, walk from P'. Connections with sequential analysis in statistics are pointed out. (Received October 9, 1950.)

48*t*. W. A. Vezeau: *On the product distribution of normally distributed variables.*

A study is made of the product, $z=xy$, where x and y are independently distributed according to the bivariate normal law. The Fourier inversion method is used to determine the frequency function, $f(z)$. For all practical purposes the function, $f(z)$, is difficult to determine. The discussion of $f(z)$ is taken up accordingly as r and s (where r and s are ratios of mean to standard deviation for x and y) are small, not so small, and large. For small r and s, $f(z)$ is expressed as an infinite set of Bessel functions, $K_n(z)$. For medium r and s, an approximation method is given with its error. Extensive calculations are made for $r=1$, $s=4$ and $r=3$, $s=5$. For larger r and s an approximation for $f(z)$ by a normal distribution is given and limits of error established. Instances of statistical problems involving the product frequency function are given. (Received October 6, 1950.)

Topology

49. M. L. Curtis: *Homotopy-regular convergence.* Preliminary report.

The notion of regular convergence of a sequence of sets as defined by G. T. Whyburn utilizes homology. By analogy one can define a regular convergence in terms of homotopy. A sequence $\{M_i\}$ of closed sets converging to a set M is said to converge *homotopy-p-regularly* provided that: Given $\epsilon>0$ there exist $\delta>0$ and an integer i_0 such that any continuous p-sphere of dia $<\delta$ in M_i, with $i>i_0$, is homotopic to a point on a set of dia $<\epsilon$ in M_i. It is shown that if M and all M_i are n-dimensional, M is LC^n, and $\{M_i\}\to M$ homotopy-p-regularly for $p=0, 1, \cdots, n$, then, for sufficiently large i, M_i has the homotopy type of M. The special case in which M is a continuum bounding a domain A in S^n is considered. In this case if the M_i separate A are LC^{q-1}, and converge homotopy-p-regularly to M for $p=0, 1, \cdots, q$, then A is ULC^q and M is LC^q. R. L. Wilder has shown that if a locally contractible continuum separates S^n and is "free in the strong sense" into complementary domain A, then A is ulc^n. Application of our last-mentioned result to this situation shows that A is ULC^n. (Received October 11, 1950.)

50*t*. M. K. Fort: *Subsets of hyperspaces and local connectivity.*

Let 2^X be the set of all nonempty compact subsets of a metric space X. Let H be the topology for 2^X which is induced by the Hausdorff metric, and let R be the topology induced by 0-regular convergence. It is shown that if A is a locally connected continuum in 2^X with respect to the R topology and each member of A is a locally connected continuum in X, then the union of the members of A is a locally connected continuum in X. An example shows that the above theorem is not true if the R topology is replaced by the H topology. (Received October 13, 1950.)

51*t*. W. S. Massey: *A new cohomology invariant of topological spaces.*

Let X be a finite simplicial complex and let H^n denote the n-dimensional cohomology group of X with coefficients in an associative ring. Choose elements $a\in H^p$ and $b\in H^q$. Define $K^m(a, b)$ to be the subgroup of H^m consisting of those elements u such that $a\cup u=0$ and $u\cup b=0$ (Alexander-Čech-Whitney product). Let $L^m(a, b) =(H^{m-q})\cup b)+(a\cup H^{m-p})$; then $L^m(a, b)$ is a subgroup of H^m. Define a homomorphism $\phi_{a,b}:K^r(a, b)\to H^s/L^s(a, b)$, where $s=p+q+r-1$, as follows. Given $u\in K^r(a, b)$, choose cocycles a', b', and u' representing a, b, and u respectively. There exist cochains x' and y' such that $a'\cup u'=\delta x'$, and $u'\cup b'=\delta y'$, where δ denotes coboundary. Then the cochain $z'=x'\cup b'+(-1)^{p-1}a'\cup y'$ is a cocycle. Let z denote its cohomology class. Define $\phi_{a,b}(u)$ to be the coset of z mod $L^s(a, b)$. The homomorphism $\phi_{a,b}$ is an invariant of the homotopy type of X, and hence, a fortiori, a topological invariant. It is possible to exhibit complexes X and Y which have isomorphic cohomology rings, no matter what the choice of coefficients, but which can be distinguished by means of this new operation. (Received September 28, 1950.)

52. E. E. Moise and G. S. Young: *On imbedding spaces in 2-manifolds.* II.

Theorem: If every point of a Peano space M has a neighborhood that can be imbedded in the plane, then M can be imbedded in some compact 2-manifold. This generalizes a previous result of the authors [Bull. Amer. Math. Soc. Abstract 54-1-82]. (Received October 11, 1950.)

53t. Deane Montgomery and Leo Zippin: *Two-dimensional subgroups.*

Let G be a separable metric connected locally compact n-dimensional group which is not compact. The authors have recently shown that G contains a closed subgroup isomorphic to the real numbers. This paper shows for n at least two that G contains a closed noncompact connected two-dimensional subgroup. (Received October 11, 1950.)

54. Jane C. Rothe: *A definition of index for certain mappings in Hilbert space.*

A topological index is defined for certain mappings of Hilbert space into itself, that is, mappings of the form $I+C+T$ where I is the identity transformation, C is linear, completely continuous, and self-adjoint and T is a higher order transformation which need not be completely continuous. This index has the usual properties of a topological index. It is shown that if T is completely continuous so that the Leray-Schauder index is defined for $I+C+T$, then the Leray-Schauder index and the index defined here are equal. Examples of mappings $I+C+T$ such that T is not completely continuous are exhibited. So it is shown that this definition of index is a nonvacuous extension of the Leray-Schauder definition. (Received October 5, 1950.)

55t. M. E. Shanks: *On semi-continuous decompositions of T_1-spaces.*

A decomposition of a T_1-space is upper semi-continuous (usc) if and only if the natural mapping on the decomposition space is closed. Dually, a similar statement is valid for lower semi-continuity (lsc) and open mappings. Different definitions of upper (lower) semi-continuity are given in terms of limits of directed sets, dusc (dlsc). The role of dusc, which differs from usc, is to preserve separation properties under the natural mapping onto the decomposition space. The two definitions of lower semi-continuity are equivalent. (Received October 11, 1950.)

J. W. T. YOUNGS,
Associate Secretary