TABLE OF THE ZEROS AND WEIGHT FACTORS OF THE FIRST FIFTEEN LAGUERRE POLYNOMIALS

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The chief use of the present table of zeros and weight factors of the Laguerre polynomials is in the performance of quadratures over the interval $[0, \infty]$, when the integrand behaves like the product of e^{-x} by a polynomial. From the well known theory of orthogonal polynomials, the quadrature formula is exact for any polynomial $P(x)$ up to the $(2n-1)$th degree, provided one uses the n-point quadrature formula. Thus if $\alpha_i^{(n)}$ denotes the "weight factors" or "Christoffel numbers," corresponding to $L_n(x)$, the nth Laguerre polynomial, and $x_i^{(n)}$ denotes the zeros of $L_n(x)$, then

$$\int_0^\infty e^{-x} P(x) dx = \sum_{i=1}^{n} \alpha_i^{(n)} P(x_i^{(n)}). \tag{1}$$

Besides problems involving direct quadratures, there are those arising in the numerical solution of linear integral equations, range $[0, \infty]$, where the unknown function occurs both inside and outside the integral sign. By considering the product of e^x by the integrand as a polynomial, and making use of (1), the approximation problem reduces to the solution of a set of only n linear equations. Hence only n points are needed to give accuracy obtainable by approximating the product of e^x by the integrand as a polynomial of the $(2n-1)$th degree. For a full description, including examples, see A. Reiz [3],[1] especially pp. 1–12.

For many purposes, the report of the Admiralty Computing Service [2], which furnishes zeros to 8 decimals and weight factors to 8 significant figures as far as $L_{10}(x)$, will suffice. This present table is intended to cope with problems requiring higher degree and accuracy. Thus there are given here the zeros and weight factors of the first fifteen Laguerre polynomials, the zeros to 12 decimals and the weight factors to 12 significant figures. (The zeros and weight factors are available in manuscript form to two extra places for $n \leqq 10$, and to one extra place for $10 < n \leqq 15$.) Also the example of A. Reiz is followed in that the quantities $\alpha_i^{(n)} e^{x_i^{(n)}}$ are also tabulated to 12 significant figures, to facilitate the quadratures when the integrand does not contain e^{-x} explicitly. Thus in

Received by the editors July 19, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

$$\int_0^\infty F(x)dx = \int_0^\infty e^{-x}[e^x F(x)]dx \sim \alpha_i^{(n)} e^{x_i^{(n)}} F(x_i^{(n)}),$$

having $\alpha_i^{(n)} e^{x_i^{(n)}}$ saves the labor of computing $e^{x_i^{(n)}}$ and then multiplying $e^{x_i^{(n)}}$ by either $\alpha_i^{(n)}$ or $F(x_i^{(n)})$. In addition, the exact values of the coefficients of the first twenty Laguerre polynomials have been computed, because of their fundamental importance.[2] They were calculated from the recursion formula (6) below, which was used to obtain $C_m^{(n+1)}$, the coefficient of x^m in $L_{n+1}(x)$, by the relation

$$C_m^{(n+1)} = (1 + 2n)C_m^{(n)} - C_{m-1}^{(n)} - n^2 C_m^{(n-1)},$$

and as a check, it was verified that

$$C_m^{(n)} = (-1)^m (n - m)!_n C_{n-m}^2.$$

The Laguerre polynomial $L_n(x)$ is defined as

$$L_n(x) = e^x \left[\frac{d^n}{dx^n} (e^{-x} x^n) \right], \tag{2}$$

from which

$$L_n(x) = (-1)^n \left(x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2(n-1)^2}{2!} x^{n-2} - \cdots \right). \tag{3}$$

The polynomial $L_n(x)$ satisfied the orthogonality relations

$$\int_0^\infty e^{-x} x^i L_n(x) dx = 0, \qquad i = 0, 1, \cdots, n - 1. \tag{4}$$

Also, $L_n(x)$ satisfies the differential equation

$$x L_n''(x) + (1 - x) L_n'(x) + n L_n(x) = 0, \tag{5}$$

and the recursion formula

$$L_{n+1}(x) - (1 + 2n - x) L_n(x) + n^2 L_{n-1}(x) = 0. \tag{6}$$

The weight factors $\alpha_i^{(n)}$ are given by

$$\alpha_i^{(n)} = \frac{1}{L_n'(x_i^{(n)})} \int_0^\infty \frac{e^{-x} L_n(x)}{x - x_i^{(n)}} dx. \tag{7}$$

The calculation of $\alpha_i^{(n)}$ is facilitated by the following relation (given in the Admiralty Report [2] and A. Reiz [3], the latter crediting it

[2] These coefficients are available in manuscript form at the Computation Laboratory.

to J. Deruyts [6]):

$$\int_0^\infty \frac{e^{-x}L_n(x)}{x - x_i^{(n)}}\,dx = \frac{(n!)^2}{x_i^{(n)}L_n'(x_i^{(n)})}, \tag{8}$$

from which there follows the very convenient formula for $\alpha_i^{(n)}$, namely,

$$\alpha_i^{(n)} = \frac{1}{x_i^{(n)}}\left[\frac{n!}{L_n'(x_i^{(n)})}\right]^2. \tag{9}$$

The zeros of $L_n(x)$ were calculated by Newton's method, from a first approximation. In most cases, for a given n, a first approximation to $x_i^{(n)}$ was obtained by either extrapolation as a function of n for the same value of i, or when i was considerably greater than 1, by extrapolation for fixed n, based upon the previous values of i (or a combination of both these processes). Then the first approximation was refined still further by 1), noting by how much previously obtained roots differed from their values obtained by this method of extrapolation, and 2), using another operation of extrapolation upon those deviations to find the deviation of the first approximation. All roots were checked by substitution into the polynomials $L_n(x)$, and noting how closely they satisfied $L_n(x_i^{(n)})=0$. In addition the roots were checked by the relations

$$\sum_{i=1}^{n} x_i^{(n)} = n^2 \tag{10}$$

and

$$\prod_{i=1}^{n} x_i^{(n)} = n!. \tag{11}$$

The first step in finding the weight factors (or Christoffel numbers) was to obtain $L_n'(x_i^{(n)})$. Because the calculations of $L_n'(x_i^{(n)})$ were lengthy, as a check upon $L_n'(x_i^{(n)})$, the second derivatives $L_n''(x_i^{(n)})$ were also computed, and were employed in the checking equation

$$x_i^{(n)}L_n''(x_i^{(n)}) = (x_i^{(n)} - 1)L_n'(x_i^{(n)}). \tag{12}$$

Then the weight factors $\alpha_i^{(n)}$ were calculated from (9) and checked by the relation $(n!)^2=\alpha_i^{(n)}x_i^{(n)}\{L_n'(x_i^{(n)})\}^2$, and also by substitution into (1), with $P(x)=x^m$, where $m!$ must be obtained. For each $L_n(x)$, m was taken successively as every number in the set 0, 1, 2, 4, 8, 16, and 24 which did not exceed $2n-1$. This wide range of m was necessary in order to be sure of including all the desired significant figures of each $\alpha_i^{(n)}$ in the checking.

The quantities $e^{x_i^{(n)}}$ were obtained by interpolation in the Mathematical Tables Project's *Tables of the exponential function* e^x and checked by interpolation in the Project's *Table of natural logarithms*, vols. III and IV. Finally, to guarantee the quantities $\alpha_i^{(n)}e^{x_i^{(n)}}$, they were checked both by division by $e^{x_i^{(n)}}$, and by duplicate calculation.

Zeros of Laguerre Polynomials	Weight Factors[3]	Weight Factors ×Exponential of Zeros
$n=1$	$n=1$	$n=1$
1.00000 00000 00	1.00000 00000 00	2.71828 18284 6
$n=2$	$n=2$	$n=2$
.58578 64376 27	.85355 33905 93	1.53332 60331 2
3.41421 35623 73	.14644 66094 07	4.45095 73350 5
$n=3$	$n=3$	$n=3$
.41577 45567 83	.71109 30099 29	1.07769 28592 7
2.29428 03602 79	.27851 77335 69	2.76214 29619 0
6.28994 50829 37	.(1) 10389 25650 16	5.60109 46254 3
$n=4$	$n=4$	$n=4$
.32254 76896 19	.60315 41043 42	.83273 91238 38
1.74576 11011 58	.35741 86924 38	2.04810 24384 5
4.53662 02969 21	.(1) 38887 90851 50	3.63114 63058 2
9.39507 09123 01	.(3) 53929 47055 61	6.48714 50844 1
$n=5$	$n=5$	$n=5$
.26356 03197 18	.52175 56105 83	.67909 40422 08
1.41340 30591 07	.39866 68110 83	1.63848 78736 0
3.59642 57710 41	.(1) 75942 44968 17	2.76944 32423 7
7.08581 00058 59	.(2) 36117 58679 92	4.31565 69009 2
12.64080 08442 76	.(4) 23369 97238 58	7.21918 63543 5
$n=6$	$n=6$	$n=6$
.22284 66041 79	.45896 46739 50	.57353 55074 23
1.18893 21016 73	.41700 08307 72	1.36925 25907 1
2.99273 63260 59	.11337 33820 74	2.26068 45933 8
5.77514 35691 05	.(1) 10399 19745 31	3.35052 45823 6

[3] The number in the parentheses stands for the number of zeros between the decimal point and the first significant figure.

Zeros of Laguerre Polynomials	Weight Factors[3]	Weight Factors ×Exponential of Zeros
$n=6$	$n=6$	$n=6$
9.83746 74183 83	.(3) 26101 72028 15	4.88682 68002 1
15.98287 39806 02	.(6) 89854 79064 30	7.84901 59456 0
$n=7$	$n=7$	$n=7$
.19304 36765 60	.40931 89517 01	.49647 75975 40
1.02666 48953 39	.42183 12778 62	1.17764 30608 6
2.56787 67449 51	.14712 63486 58	1.91824 97816 6
4.90035 30845 26	.(1) 20633 51446 87	2.77184 86362 3
8.18215 34445 63	.(2) 10740 10143 28	3.84124 91224 9
12.73418 02917 98	.(4) 15865 46434 86	5.38067 82079 2
19.39572 78622 63	.(7) 31703 15479 00	8.40543 24868 3
$n=8$	$n=8$	$n=8$
.17027 96323 05	.36918 85893 42	.43772 34104 93
.90370 17767 99	.41878 67808 14	1.03386 93476 7
2.25108 66298 66	.17579 49866 37	1.66970 97656 6
4.26670 01702 88	.(1) 33343 49226 12	2.37692 47017 6
7.04590 54023 93	.(2) 27945 36235 23	3.20854 09133 5
10.75851 60101 81	.(4) 90765 08773 36	4.26857 55108 3
15.74067 86412 78	.(6) 84857 46716 27	5.81808 33686 7
22.86313 17368 89	.(8) 10480 01174 87	8.90622 62152 9
$n=9$	$n=9$	$n=9$
.15232 22277 32	.33612 64217 98	.39143 11243 16
.80722 00227 42	.41121 39804 24	.92180 50285 29
2.00513 51556 19	.19928 75253 71	1.48012 79099 4
3.78347 39733 31	.(1) 47460 56276 57	2.08677 08075 5
6.20495 67778 77	.(2) 55996 26610 79	2.77292 13897 1
9.37298 52516 88	.(3) 30524 97670 93	3.59162 60680 9
13.46623 69110 92	.(5) 65921 23026 08	4.64876 60021 4
18.83359 77889 92	.(7) 41107 69330 35	6.21227 54197 5
26.37407 18909 27	.(10) 32908 74030 35	9.36321 82377 1
$n=10$	$n=10$	$n=10$
.13779 34705 40	.30844 11157 65	.35400 97386 07
.72945 45495 03	.40111 99291 55	.83190 23010 44

Zeros of Laguerre Polynomials	Weight Factors[3]	Weight Factors ×Exponential of Zeros
$n=10$	$n=10$	$n=10$
1.80834 29017 40	.21806 82876 12	1.33028 85617 5
3.40143 36978 55	.(1) 62087 45609 87	1.86306 39031 1
5.55249 61400 64	.(2) 95015 16975 18	2.45025 55580 8
8.33015 27467 64	.(3) 75300 83885 88	3.12276 41551 4
11.84378 58379 00	.(4) 28259 23349 60	3.93415 26955 6
16.27925 78313 78	.(6) 42493 13984 96	4.99241 48721 9
21.99658 58119 81	.(8) 18395 64823 98	6.57220 24851 3
29.92069 70122 74	.(12) 99118 27219 61	9.78469 58403 7
$n=11$	$n=11$	$n=11$
.12579 64421 88	.28493 32128 94	.32312 88804 35
.66541 82558 39	.38972 08895 28	.75812 55998 10
1.64715 05458 72	.23278 18318 49	1.20864 14229 0
3.09113 81430 35	.(1) 76564 45354 62	1.68457 91671 4
5.02928 44015 80	.(1) 14393 28276 74	2.19963 34786 2
7.50988 78638 07	.(2) 15188 80846 48	2.77348 97457 5
10.60595 09995 47	.(4) 85131 22435 47	3.43712 14165 2
14.43161 37580 64	.(5) 22924 03879 57	4.24484 02908 0
19.17885 74032 15	.(7) 24863 53702 77	5.30682 60194 8
25.21770 93396 78	.(10) 77126 26933 69	6.90421 54788 3
33.49719 28471 76	.(13) 28837 75868 32	10.17671 27469
$n=12$	$n=12$	$n=12$
.11572 21173 58	.26473 13710 55	.29720 96360 44
.61175 74845 15	.37775 92758 73	.69646 29804 31
1.51261 02697 76	.24408 20113 20	1.10778 13946 2
2.83375 13377 44	.(1) 90449 22221 17	1.53846 42390 4
4.59922 76394 18	.(1) 20102 38115 46	1.99832 76062 7
6.84452 54531 15	.(2) 26639 73541 87	2.50074 57691 0
9.62131 68424 57	.(3) 20323 15926 63	3.06532 15182 8
13.00605 49933 06	.(5) 83650 55856 82	3.72328 91107 8
17.11685 51874 62	.(6) 16684 93876 54	4.52981 40299 8
22.15109 03793 97	.(8) 13423 91030 52	5.59725 84618 4
28.48796 72509 84	.(11) 30616 01635 04	7.21299 54609 3
37.09912 10444 67	.(15) 81480 77467 43	10.54383 74619

Zeros of Laguerre Polynomials	Weight Factors[3]	Weight Factors ×Exponential of Zeros
$n=13$	$n=13$	$n=13$
.10714 23884 72	.24718 87084 30	.27514 39554 71
.56613 18990 40	.36568 88229 01	.64413 90765 43
1.39856 43364 51	.25256 24200 58	1.02272 17785 8
2.61659 71084 06	.10347 07580 24	1.41641 76056 6
4.23884 59290 17	.(1) 26432 75441 56	1.83252 46178 6
6.29225 62711 40	.(2) 42203 96040 27	2.28058 03822 9
8.81500 19411 87	.(3) 41188 17704 73	2.77382 63658 6
11.86140 35888 11	.(4) 23515 47398 15	3.33192 99564 3
15.51076 20377 04	.(6) 73173 11620 25	3.98648 29749 2
19.88463 56638 80	.(7) 11088 41625 70	4.79354 78219 3
25.18526 38646 78	.(10) 67708 26692 21	5.86763 13371 1
31.80038 63019 47	.(12) 11599 79959 91	7.50209 88049 2
40.72300 86692 66	.(16) 22450 93203 89	10.88961 00139
$n=14$	$n=14$	$n=14$
.09974 75070 33	.23181 55771 45	.25613 11547 37
.52685 76488 52	.35378 46915 98	.59917 04774 95
1.30062 91212 51	.25873 46102 45	.94997 15024 75
2.43080 10787 31	.11548 28935 57	1.31280 78114 7
3.93210 28222 93	.(1) 33192 09215 93	1.69326 59907 5
5.82553 62183 02	.(2) 61928 69437 01	2.09840 90842 7
8.14024 01415 65	.(3) 73989 03778 67	2.53763 26549 4
10.91649 95073 66	.(4) 54907 19466 84	3.02415 99177 0
14.21080 50111 61	.(5) 24095 85764 09	3.57780 43557 2
18.10489 22202 18	.(7) 58015 43981 68	4.23056 50349 1
22.72338 16282 69	.(9) 68193 14692 49	5.03941 33684 6
28.27298 17232 48	.(11) 32212 07751 89	6.12094 79283 3
35.14944 36605 92	.(14) 42213 52440 52	7.77429 53670 1
44.36608 17111 17	.(18) 60523 75022 29	11.21683 42167
$n=15$	$n=15$	$n=15$
.09330 78120 17	.21823 48859 40	.23957 81703 11
.49269 17403 02	.34221 01779 23	.56010 08427 93
1.21559 54120 71	.26302 75779 42	.88700 82629 19
2.26994 95262 04	.12642 58181 06	1.22366 44021 5
3.66762 27217 51	.(1) 40206 86492 10	1.57444 87216 3
5.42533 66274 14	.(2) 85638 77803 61	1.94475 19765 3

Zeros of Laguerre Polynomials	Weight Factors[3]	Weight Factors ×Exponential of Zeros
$n=15$	$n=15$	$n=15$
7.56591 62266 13	.(2) 12124 36147 21	2.34150 20566 4
10.12022 85680 19	.(3) 11167 43923 44	2.77404 19268 3
13.13028 24821 76	.(5) 64599 26762 02	3.25564 33464 0
16.65440 77083 30	.(6) 22263 16907 10	3.80631 17142 3
20.77647 88994 49	.(8) 42274 30384 98	4.45847 77538 4
25.62389 42267 29	.(10) 39218 97267 04	5.27001 77844 3
31.40751 91697 54	.(12) 14565 15264 07	6.35956 34697 3
38.53068 33064 86	.(15) 14830 27051 11	8.03178 76321 2
48.02608 55726 86	.(19) 16005 94906 21	11.52777 21009

BIBLIOGRAPHY

NOTE: A detailed bibliography on Laguerre polynomials is really unnecessary, since a very exhaustive one up to the year 1940 is given in J. Shohat [1]. However, the few references given below afford a representative sample and cover many essential aspects of Laguerre polynomials.

1. J. A. Shohat, *A bibliography on orthogonal polynomials*, Bulletin of the National Research Council, no. 103, 1940. Lists over 2000 articles, books, and theses, by 643 authors, on the subject of orthogonal polynomials. The topical index has a special section on Laguerre polynomials, where more than 250 references are given.

2. Department of Scientific Research and Experiment, Admiralty Computing Service, *Zeros of Laguerre polynomials and the corresponding Christoffel numbers*, June, 1945.

3. A. Reiz, *On the numerical solution of certain types of integral equations*, Arkiv för Matematik, Astronomi och Fysik, vol. 29A, no. 29.

4. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939. Includes a large bibliography.

5. D. Jackson, *Fourier series and orthogonal polynomials*, Carus Mathematical Monographs, no. 6, Mathematical Association of America, 1941. Chapters VII (general) and X (on Laguerre polynomials).

6. J. Deruyts, *Sur le calcul approché de certaines intégrales définies*, Bulletins de l'Académie Royale de Belgique (3) vol. 11 (1886) pp. 307–311.

7. W. Hahn, *Bericht über die Nullstellen der Laguerreschen und der Hermiteschen Polynome*, Jber. Deutschen Math. Verein. vol. 44 (1934) pp. 215–236. An elaborate bibliography is included in this paper.

8. E. Moecklin, *Asymptotische Entwicklungen der Laguerreschen Polynome*, Comment. Math. Helv. vol. 7 (1934) pp. 24–46.

9. F. Tricomi, *Generalizzazione di una formula asintotica sui polinomi di Laguerre e sue applicazioni*, Atti Accad. Sci. Torino vol. 76 (1941) pp. 288–316.

10. E. C. Kemble, *The fundamental principles of quantum mechanics*, McGraw-Hill, 1937, pp. 160, 585–587.

11. J. Cossar and A. Erdélyi, *Dictionary of Laplace transforms*, Part 2B (issued by Dept. of Scientific Research and Experiment, Admiralty Computing Service), pp. 84–85.

12. H. Margenau and G. M. Murphy, *The mathematics of physics and chemistry*, New York, Van Nostrand, 1943, pp. 77–78, 122–128, 350.

13. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. I, Berlin, Springer, 1931, pp. 79–80, 284–285, 294–296, 324–326, 440–441. (Reprinted by Interscience Publishers, 1943.)

14. N. Košliakov, *The calculation by a formula of mechanical quadrature of definite integrals with infinite limits* (Russian), Bull. Acad. Sci. URSS. (7) (1933) pp. 801–808.

Of the above references, [2], [3], and [14] include zeros and Christoffel numbers. Those in [2] agree exactly with the table above as far as comparison can be made. The zeros and Christoffel numbers in [3], both given to $7D$, up to $n=5$, also agree with this table as far as comparison can be made, save for a few values that differ by no more than 2 units in the last place. The zeros and Christoffel numbers in [14], given only for $n=5$, to $7D$, are all incorrect (except for A_4) up to as much as 63 units in the 7th decimal place. For details on [14], see MTAC III, no 21, Jan. 1948, item 121 on p. 41. In a footnote to [9], there are given 5-decimal values of the zeros of the first few Laguerre polynomials, correct to within several units of the last place. But in the text on p. 308, $x_{10}^{(10)}$ is erroneously given as 29.9*315* instead of 29.9*207*. In [3], there occurs $\alpha_i^{(n)}e^{x_i^{(n)}}$, up to $n=5$, to $7D$.

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