INTEGRAL DISTANCES IN BANACH SPACES

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In a recent paper,[1] Anning and Erdös have shown that in a finite-dimensional euclidean space a necessary and sufficient condition that an infinite set of points have the property that the distance between any two of them be an integer is that the points lie on a straight line. In this paper we shall investigate to what extent this theorem is true for general normed linear spaces of finite or infinite dimension. The theorem as stated is evidently not true for spaces of infinite dimension since in the space l^2 the infinite set of elements $z_i=(0, 0, \cdots, 2^{1/2}/2, 0, \cdots)$ with $2^{1/2}/2$ in the ith place and zero elsewhere have the property that $\|z_i-z_j\|=1$ for $i\neq j$ and the z_i are linearly independent and not collinear. We shall prove that if a Banach space is not strictly convex there exists an infinite set of points each at an integral distance from the others which do not lie on a straight line. If the space is strictly convex, any such set has the property that if an infinite number of its points lie on a straight line then all the points of the set lie on the line. It is possible to prove certain theorems in analysis by applying these theorems to function spaces.

We shall define a *minimal set* connecting the points x_1 and x_2 in a Banach space X to be the set of all points of the space for which $\|x_1-x\|+\|x_2-x\|=\|x_1-x_2\|$. A *straight line* containing x_1 and x_2 is the set of all points of the form $\alpha x_1+\beta x_2$ where $\alpha+\beta=1$. The set of points of the line for which α and β are non-negative is the *line segment* joining x_1 and x_2. If $\alpha+\beta=1, \alpha\geqq 0, \beta\geqq 0$

$$\begin{aligned}\|x_1 - (\alpha x_1 + \beta x_2\| &+ \|x_2 - (\alpha x_1 + \beta x_2)\| \\ &= \|(1-\alpha)x_1 - \beta x_2\| + \|(1-\beta)x_2 - \alpha x_1\| \\ &= \|\beta x_1 - \beta x_2\| + \|\alpha x_2 - \alpha x_1\| = \beta\|x_2 - x_1\| + \alpha\|x_2 - x_1\| \\ &= (\alpha+\beta)\|x_2 - x_1\| = \|x_2 - x_1\|\end{aligned}$$

and the line segment joining x_1 and x_2 is in the minimal set determined by x_1 and x_2. A space in which the minimal set consists only of the line segment for any pair of points is called a *straight line space.* We shall show that such spaces are the strictly convex spaces.

THEOREM 1. *A necessary and sufficient condition that the minimal*

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[1] N. A. Anning and P. Erdös, *Integral distances*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 598–600.

set between any two points of a Banach space consist only of the line segment joining the points is that the space be strictly convex.

PROOF. Suppose that the unit sphere of X is not strictly convex and that there exist points x_1 and x_2, $\|x_1\|=\|x_2\|=1$, and $\|\alpha x_1+\beta x_2\|=1$, $\alpha+\beta=1$, α, $\beta\geqq 0$. Then the line segment joining $-x_1$ and $-x_2$ is also on the boundary of the unit sphere. If the sphere is translated to the point x_1+x_2 then $(x_1+x_2)-x_2=x_1$ and $(x_1+x_2)-x_1=x_2$ and the line segment joining $-x_1$ and $-x_2$ is translated to coincide with the segment joining x_2 and x_1. Hence the distance between x_1+x_2 and a point on the segment is one, $\|(x_1+x_2)-(\alpha x_1+\beta x_2)\|=1$. Also, $\|\alpha x_1+\beta x_2\|=1$. Thus

$$\|(x_1 + x_2) - (\alpha x_1 + \beta x_2)\| + \|\alpha x_1 + \beta x_2\| = \|x_1 + x_2\|$$

and the segment $\alpha x_1+\beta x_2$ is in the minimal set between θ and x_1+x_2. Hence the minimal set contains points other than those on the segment between θ and x_1+x_2.

Let X be strictly convex and let M be the minimal set between θ and a point x_1. Consider the points $y\in M$ such that $\|y\|=d<\|x_1\|$. One such point y_0 is on the segment γx, $0\leqq\gamma\leqq 1$. If there were another such point y_1, the line segment between y_0 and y_1 would be interior to both the spheres $\|x-x_1\|\leqq\|x_1\|-d$ and $\|x\|\leqq d$ since both these spheres are strictly convex. If y is a point of this segment, $y\neq y_0$, y_1, $\|x_1-y\|+\|y\|<\|x_1\|-d+d=\|x_1\|$ which is not possible. This shows that y_0 is the only point of M at a given distance d from θ. Since the same argument may be used for any pair of points in X, X is a straight line space.

The above proof shows in particular that in a strictly convex space, metric definitions of a straight line coincide with algebraic definitions.

THEOREM 2. *If X is a Banach space which is not strictly convex there exists an infinite set of points $\{x_i\}$ such that $\|x_j-x_k\|$ is an integer for all j and k and the points are not all on the same straight line.*

PROOF.[2] Let S be the unit sphere in X, let $y_0=0$ and let $\{y_i\}$, $i=1, 2, 3, \cdots$, be a set of points on the boundary of S which all lie on the same line segment L. Let $x_n=\sum_{i=0}^{n} y_i$, $n=1, 2, 3, \cdots$. Evidently $\|x_0\|=0$, $\|x_1\|=1$, and $\|x_2\|=\|y_1+y_2\|=2\|(y_1+y_2)/2\|=2$ since $\|x_1/2+x_2/2\|=1$. Assume that $\|x_{n-1}\|=n-1$ and that $(1/(n-1))\cdot x_{n-1}\in L$. Then

$$\|x_n\| = \left\|\sum_{i=0}^{n} y_i\right\| = \|x_{n-1} + y_n\|$$

[2] This proof was suggested by the referee.

$$= n\left\|\frac{n-1}{n}\frac{1}{n-1}x_{n-1}+\frac{1}{n}y_n\right\| = n.$$

Hence $\|x_n\|$ is an integer and $(1/n)x_n \in L$. Since the above proof holds for any set y_i of L, we have

$$\|x_n - x_m\| = \left\|\sum_{i=m}^{n} y_i\right\| = \left\|y_0 + \sum_{i=m}^{n} y_i\right\| = |n-m|$$

and the distance between any two points of the set is an integer. The points do not all lie on the same line since in this case they would be of the form ny_i and $y_i = y_1$ for each i.

In case the set $\{y_i\}$ is a finite set, $i = 1, 2, \cdots, k$, the same procedure can be followed and we can define the points $y_{i+mk} = y_i$ for each i between 0 and k. For $n \geqq k$ the points x_n would be of the form

$$x_n = \sum_{i=0}^{m}(j+1)y_i + \sum_{i=m+1}^{k} jy_i = j\sum_{i=0}^{k} y_i + \sum_{i=0}^{m} y_i = ju + v_m$$

where $n = jk + m$ and $u = \sum_{i=0}^{k} y_i$, $v_m = \sum_{i=0}^{m} y_i$. Thus all points of the set in this case lie on the k lines $\gamma u + v_m$, $m = 1, 2, \cdots, k$, which are parallel in the sense that they are all translates of the line γu.

THEOREM 3. *Let the space X be strictly convex. Let $\{x_i\}$ be an infinite set of points of X such that $\|x_k - x_j\|$ is an integer for each j and k. Then if any infinite subset of this set consists of collinear points, all the points of the set are collinear.*

In the proof of this theorem we make use of two lemmas.

LEMMA 1. *Let X be strictly convex and let S_1 and S_2 be two spheres of different radii, r_1 and r_2, with centers x_1, x_2. Let x_3 be a point common to the boundaries of the two spheres lying on the line joining the two centers. Then x_3 is the only point common to the boundary of the two spheres and, except for x_3, S_1 is entirely interior to S_2, S_2 is interior to S_1, or the two spheres are entirely exterior to each other.*

PROOF. Suppose that x_4 is another point common to the boundaries of the two spheres. If x_4 is distinct from x_3, x_4 is not on the line of centers of the two spheres. If x_1 lies between x_2 and x_3 we have

$$\begin{aligned} r_2 &= \|x_3 - x_2\| = \|x_3 - x_1\| + \|x_1 - x_2\| = r_1 + \|x_1 - x_2\| \\ &= \|x_4 - x_1\| + \|x_1 - x_2\|. \end{aligned}$$

However, $r_2 = \|x_4 - x_2\|$ and this would imply that x_4 lies on the line of centers of the two spheres and gives a contradiction. In case x_3

lies between x_1 and x_2

$$\begin{aligned}\|x_1 - x_2\| &= \|x_1 - x_3\| + \|x_2 - x_3\| = r_1 + r_2\\ &= \|x_1 - x_4\| + \|x_2 - x_4\|\end{aligned}$$

which implies again that x_4 lies on the line between x_1 and x_2. A similar argument holds if x_2 is between x_1 and x_3 and hence x_3 is the only point common to the two spheres.

To prove the second part of the lemma assume $r_1 \leqq r_2$, $x_1 \neq x_2$. Then either $\|x_1 - x_3\| + \|x_3 - x_2\| = \|x_1 - x_2\|$ or $\|x_1 - x_2\| + \|x_2 - x_3\| = \|x_1 - x_3\|$. Let x be any point of S_2. In the first case

$$\begin{aligned}\|x_1 - x\| + \|x_2 - x\| &\geqq \|x_1 - x_2\|\\ &= \|x_1 - x_3\| + \|x_3 - x_2\| = r_1 + r_2\end{aligned}$$

and $\|x_1 - x\| > r_1$ if $x \neq x_3$. Thus S_2 is entirely exterior to S_1 except for x_3. By a similar method it can be seen that in the second case S_2 is interior to S_1.

LEMMA 2. *Let a be a point of X and let r be a positive number $r < \|a\|$. Let H be a "hyperboloid" in X with foci at θ and a, that is, the set of points $x \in X$ such that $\|x\| - \|x - a\| = r$. Then a line through the focus θ has at most one point in common with H and a line through a has at most two points in common with H.*

PROOF. Let S_0 be the sphere with radius r and center θ. The set H is the locus of the centers of all spheres exterior to S_0 containing a on their boundaries and having one point in common with S_0. Let l be a line through θ and suppose that l has a point x_0 in common with H. Let $S(x_0)$ be the sphere exterior to S_0 with center x_0 having one point y in common with S_0 and containing a on its boundary. Evidently l contains the point y. Suppose that l intersects H in another point x_1. Then $S(x_1)$ must pass through y and hence must be exterior to $S(x_0)$, contain $S(x_0)$, or be contained in $S(x_0)$. In the first case y is exterior to the segment joining 0 and x_1 and hence $S(x_1)$ is not exterior to S_0. In the other two cases $S(x_1)$ cannot contain a on its boundary and hence x_1 cannot be in H.

Let l be a line through a. Suppose l intersects H in a point x_0. Then $S(x_0)$ has a on its boundary and also a point of S_0. If l intersects H in another point x_1, $S(x_1)$ contains $S(x_0)$, is contained in $S(x_0)$, or is exterior to $S(x_0)$. An argument similar to that of the first part of the proof shows that the first two cases cannot occur. Suppose that the third case occurs and that l intersects H in a third point x_2. Then $S(x_0)$, $S(x_1)$, and $S(x_2)$ all pass through a. By using Lemma 1 it can be seen that if x_0, x_1, and x_2 are distinct then $S(x_2)$ must either contain or be contained in $S(x_0)$ or $S(x_1)$. In any case it cannot have only one point in common with S_0 and hence l has at most two points in common with H.

It is evident that the above lemma holds for H defined by any two foci.

PROOF OF THE THEOREM. Suppose that an infinite set of the x_i lie on a line l. It can be assumed that $x_1=\theta$. Let a be a point of the set not on l. Consider the family of hyperboloids $\|x-a\|-\|x\|=n$ and $\|x\|-\|x-a\|=n$ for n an integer or zero. In the first case $\|a\|+\|x\| \geqq \|x-a\|=n+\|x\|$ and in the second case $\|x\|=n+\|x-a\|\leqq\|a\| +\|x-a\|$ and in both cases if the hyperboloid is to contain points not on the line between θ and a, n cannot exceed $\|a\|-1$. Since for each i, $\|x_i\|$ and $\|x_i-a\|$ are integers, $\|x_i-a\|-\|x_i\|$ or $\|x_i\|-\|x_i-a\|$ is an integer. Since l can intersect each of the first type of hyperboloids in at most two points and each of the second in at most one point, l cannot contain more than $3\|a\|-1$ points of the set. Hence if l contains an infinite subset of the given set it must contain all points of the set.

The above proof shows that if X is strictly convex and if x_i and x_j are two points of the set then any line through x_i but not through x_j contains not more than $3\|x_i-x_j\|-1$ points of the set. If the norm of the difference is one the integer n of the hyperboloids described above is zero and the only hyperboloid between the two points is the bisecting hyperplane of the segment joining the two points. Hence the line through x_i but not x_j can contain at most one other point. Applied to the set z_i in l^2 discussed at the beginning of the paper this shows that a line through any point of the set contains at most one other point of the set.

The above theorems give interesting results when applied to various well known strictly convex spaces. For example, let $\{x_i(t)\}$ be a set of functions in the space $L^p(a, b)$, $p>1$, such that $x_0(t)\equiv 0$ and $(\int_a^b |x_i(t)-x_j(t)|^p dt)^{1/p}$ is an integer for each i and j. The following corollaries are true.

COROLLARY 1. *Let the family $\{x_i(t)\}$ contain an infinite subfamily of the form $\alpha_j x_1(t)$. Then every function of the family is a multiple of $x_1(t)$.*

COROLLARY 2. *For any k, the number of multiples of $x_k(t)$ which may occur in the set cannot exceed $3\cdot\min(\int_a^b|x_k(t)-x_j(t)|^p dt)^{1/p}-1$ where the minimum is taken for $j\neq k$ and $x_j(t)$ not a multiple of $x_k(t)$.*

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