# A METHOD OF ANALYTIC CONTINUATION SUGGESTED BY HEURISTIC PRINCIPLES

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Suppose we are given an analytic function $f(z)$ represented by a power series (supposed convergent for some $z \neq 0$)

$$f(z) = \sum_{n=0}^{\infty} \frac{f_n}{n!} z^n \tag{1}$$

where $f_n$ is the value of the $n$th derivative of $f(z)$ at the origin. For small values of $\delta$ we can approximate $f(\delta)$ in the following manner:

$$f(\delta) \simeq f_0 + \delta f_1. \tag{2}$$

Refinement of this approximation leads to Taylor's theorem and back to the power series (1). It is possible, however, to use the linear approximation in a different way: we can use such approximations to go from one point to another along a chain of points $z = \delta$, $2\delta$, $3\delta, \cdots, n\delta$. Thus we shall say

$$f(\delta) \simeq f_0 + \delta f_1, \qquad f'(z)\big|_{z=\delta} \simeq f_1 + \delta f_2,$$

and so on, and

$$f(2\delta) \simeq f(\delta) + \delta f'(z)\big|_{z=\delta} \simeq f_0 + 2\delta f_1 + \delta^2 f_2.$$

In general

$$f(n\delta) \simeq \sum_{m=0}^{n} a_{n,m} f_m \delta^m. \tag{3}$$

It is easily verified that $a_{n,m} = a_{n-1,m} + a_{n-1,m-1}$ and hence that $a_{n,m}$ is the binomial coefficient $C_{n,m}$. If we now define the following:

$$\sigma_n(z) = \sum_{m=0}^{n} f_m C_{n,m} \left(\frac{z}{n}\right)^m \tag{4}$$

and if $n\delta = z$, then (3) is equivalent to

$$f(z) \simeq \sigma_n(z). \tag{5}$$

The question now presents itself: for what values of $z$ does the sequence of polynomials $\sigma_n(z)$ converge to the function $f(z)$? It is evident that if $z$ is inside the circle of convergence of the series (1), then

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we may say directly

$$\begin{aligned}\lim_{n\to\infty}\sigma_n(z) &= \lim_{n\to\infty}\sum_{m=0}^{n} f_m C_{n,m}\left(\frac{z}{n}\right)^m\\ &= \lim_{n\to\infty}\sum_{m=0}^{n}\frac{f_m z^m}{m!}\;\frac{n!}{(n-m)!n^m}\\ &= \sum_{m=0}^{\infty}\frac{f_m z^m}{m!} = f(z).\end{aligned}\tag{6}$$

However, this is not the complete answer.

It was shown by Borel[2] that we can without loss of generality restrict ourselves to the special case $f(z)=1/(1-z)$. Here

$$f_n = n!,$$

$$\sigma_n(z) = \sum_{m=0}^{n}\frac{n!}{(n-m)!}\left(\frac{z}{n}\right)^m.\tag{7}$$

Obviously if $|z|<1$

$$\lim_{n\to\infty}\sigma_n(z) = \sum_{m=0}^{\infty} z^m = \frac{1}{1-z}\tag{8}$$

and the convergence is uniform if $|z|\leqq 1-\epsilon$. It will be deduced later that the same is true if $|z|\leqq 1$ except for a neighborhood of the point $z=1$. Hence we shall confine ourselves to values of $z$ such that $|z|>1$. From (7)

$$\begin{aligned}\sigma_n(z) &= \sum_{m=0}^{n}\frac{n!}{(n-m)!}\left(\frac{z}{n}\right)^m\\ &= n!\left(\frac{z}{n}\right)^n\sum_{p=0}^{n}\frac{1}{p!}\left(\frac{n}{z}\right)^p\\ &= n!\left(\frac{z}{n}\right)^n e^{n/z} - \sum_{p=n+1}^{\infty}\frac{n!}{p!}\left(\frac{n}{z}\right)^{p-n},\end{aligned}$$

$$\sigma_n(z) = n!\left(\frac{z}{n}\right)^n e^{n/z} - \sum_{r=1}^{\infty} z^{-r}\frac{n!n^r}{(n+r)!}.\tag{9}$$

If

$$|z| > 1\tag{10}$$

[2] Borel, *Leçons sur les series divergentes*, Paris, 1924, pp. 198 ff.

then

$$\lim_{n\to\infty} -\sum_{r=1}^{\infty} z^{-r}\frac{n!n^r}{(n+r)!} = -\sum_{r=1}^{\infty} z^{-r} = \frac{1}{1-z} \tag{11}$$

and the convergence is uniform if $|z| \geqq 1+\epsilon$. Using Stirling's formula, we have

$$\lim_{n\to\infty} n!\left(\frac{z}{n}\right)^n e^{n/z} = \lim_{n\to\infty} (2\pi n)^{1/2}(ze^{-1+1/z})^n. \tag{12}$$

This is zero if

$$|ze^{-1+1/z}| < 1. \tag{13}$$

If we let $z=\rho e^{i\theta}$ then

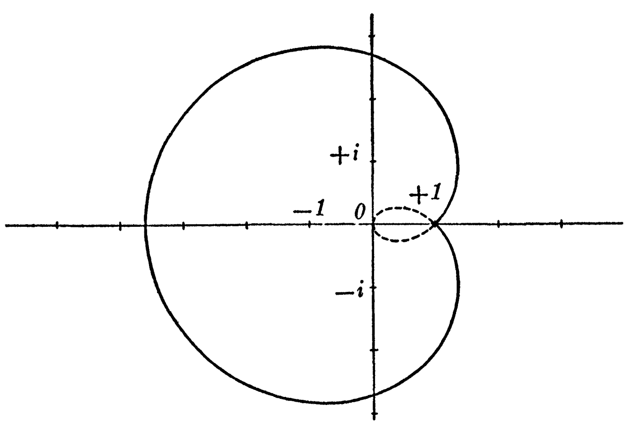
$$|ze^{-1+1/z}| = \rho e^{-1+(\cos\theta)/\rho}$$

and (13) is the same as

$$\rho e^{-1+(\cos\theta)/\rho} < 1$$

or

$$\cos\theta < \rho(1-\log\rho). \tag{14}$$




The curve $\cos\theta=\rho(1-\log\rho)$ and the region $S$. The region $S$ is bounded by the solid part of the curve.

We shall call the region bounded by the curve $\cos\theta=\rho(1-\log\rho)$, $\rho\geqq 1$, the region $S$ (see figure). In view of (8)–(12) and (14) we see that this is the region in which $\sigma_n(z)\to 1/(1-z)$ uniformly, except possibly for a neighborhood of $|z|=1$ and a neighborhood of the boundary.

For the purpose of applying Borel's theorems it is also necessary to show that in the region $(1-\epsilon\leqq\rho\leqq 1+\epsilon,\ \epsilon'\leqq\theta\leqq 2\pi-\epsilon')$, $\sigma_n(z)$ is bounded uniformly in $z$. This follows for $\rho\leqq 1$ from (7), by an application of Abel's transformation to the series $\sum_0^\infty z^n$ with the coefficients (obviously monotonically decreasing) $n!/(n-m)!n^m$. Similarly it follows from (9) for $\rho\geqq 1$.

Borel[3] now proceeds as follows: suppose we have a system of polynomials $\sigma_n(z)=c_0^{(n)}+c_1^{(n)}z+c_2^{(n)}z^2+\cdots$ and such that $\sigma_n(z)\to 1/(1-z)$. Suppose we have also a function $f(z)$ represented by the power series

$$\sum_{n=0}^{\infty}\frac{f_n}{n!}z^n.$$

We may write Cauchy's integral thus:

$$f(z)=\frac{1}{2\pi i}\int_c\frac{f(x)}{x}\,\frac{1}{1-z/x}\,dx$$

where $c$ is a contour around $z$ not including any singularities of $f(z)$. Let

$$F_n(z)=\frac{1}{2\pi i}\int_c\frac{f(x)}{x}\,\sigma_n\left(\frac{z}{x}\right)dx.$$

Borel showed that

$$F_n(z)=\sum_{k=0}^{\infty}c_k^{(n)}\frac{f_k}{k!}z^k.$$

In case $\sigma_n(z)\to 1/(1-z)$ uniformly in a certain star-shaped region including the circle of convergence of $\sum_0^\infty z^n$ Borel deduces by appropriate choice of the contour $c$ a region in which $F_n(z)\to f(z)$ which we describe as follows:

Construct for each singularity $\zeta$ of $f(z)$ a region $R(\zeta)$ similar to that in which $\sigma_n(z)$ converges to $1/(1-z)$, such that the point corresponding to the origin is at the origin and the point corresponding to $z=1$ is at the singularity $\zeta$. Let $R=\prod_\zeta R(\zeta)$ be the region common to all the $R(\zeta)$. In any finite region $R'$ interior to $R$

$$F_n(z)\to f(z)$$

and the convergence is uniform if it is uniform in the special case

[3] Borel, loc. cit. We have changed the notation to agree with our own.

$f(z)=1/(1-z)$, $F_n(z)=\sigma_n(z)$.

In our case, $c_k^{(n)}=n!/(n-k)!n^k$, $\sigma_n(z)\rightarrow 1/(1-z)$ uniformly in any subregion of $S$ interior to $S$ except in a region as small as we please where $|\sigma_n(z)-1/(1-z)|$ is uniformly bounded. The reader may convince himself that this restriction does not alter Borel's results.

If in the identity (9) we replace $z$ by $1/z$ we can derive a more powerful method of continuation. It is not, however, as powerful as those of Borel, Mittag-Leffler, and others, and hence we shall not discuss it.

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