AN INTEGRATION SCHEME OF MARÉCHAL

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The French physicist Maréchal [1][1] has invented a mechanical integrator for studying the distribution of light in an optical image. This integrator approximates a double integral $\int_0^{2\pi}\int_0^R f(r, \phi)rdrd\phi$ by a line integral $2\pi a\int_C f(r, \phi)ds$ extended over that portion of an archimedean spiral

$$C: \qquad r = a\phi$$

which lies inside the circle $0 \leqq r \leqq R$, $0 \leqq \phi < 2\pi$. The validity of this procedure when $f(r, \phi)$ is continuous (as it always is in the case of the integrals determining distribution of light in an optical image) was taken for granted by Maréchal when a is small. It is the purpose of this note to justify Maréchal's approximation by proving the following theorem.

THEOREM. *If $f(r, \phi)$ is continuous on $0 \leqq r \leqq R$, $0 \leqq \phi < 2\pi$ and is periodic with period 2π in ϕ, then*

$$\lim_{a=0} 2\pi \int_0^R f(r, r/a)(a^2 + r^2)^{1/2}dr = \int_0^{2\pi}\int_0^R f(r, \phi)rdrd\phi. \tag{1}$$

Let us define

$$\begin{aligned} P_N(r, \phi) &= \frac{1}{2N\pi}\int_0^{\pi} \{f(r, \phi + u) \\ &\qquad + f(r, \phi - u)\} \sin^2 (Nu/2) \csc^2 (u/2)du, \\ a_{nN}(r) &= \frac{1}{2\pi}(1 - |n|/N)\int_0^{2\pi} f(r, \phi)e^{-in\phi}d\phi. \end{aligned} \tag{2}$$

Then it is known from the theory of $(C, 1)$ summability of Fourier series that

$$\begin{aligned} P_N(r, \phi) &= \sum_{n=-(N-1)}^{n=N-1} a_{nN}(r)e^{in\phi}, \\ \lim_{N=\infty} P_N(r, \phi) &= f(r, \phi) \end{aligned} \tag{3}$$

uniformly on $0 \leqq r \leqq R$, $0 \leqq \phi < 2\pi$. For each positive ϵ we can therefore

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[1] Numbers in brackets refer to the reference cited at the end of the paper.

pick an integer N such that

$$(4) \qquad |P_N(r,\phi) - f(r,\phi)| < \epsilon/3\pi R^2.$$

Let M be the maximum of $|f(r,\phi)|$. Then

$$\left| 2\pi\int_0^R f(r, r/a)(a^2+r^2)^{1/2}dr - 2\pi\int_0^R f(r,r/a)rdr \right| \leqq 2\pi M\int_0^R \left\{(a^2+r^2)^{1/2} - r\right\}dr < \epsilon/3$$

if a is less than a suitably chosen $A_1(\epsilon)$. Moreover, since (4) holds

$$\left| 2\pi\int_0^R f(r,r/a)rdr - 2\pi\int_0^R P_N(r,r/a)rdr \right| < \epsilon/3.$$

By virtue of (3),

$$\left| 2\pi\int_0^R P_N(r,r/a)rdr - 2\pi\int_0^R a_{0N}(r)rdr \right| \leqq 2\pi \sum_{|n|=1}^{N-1} \left| \int_0^R a_{nN}(r)e^{inr/a}dr \right|.$$

Since N is fixed when ϵ is chosen we infer from the Riemann-Lebesgue lemma (which is surely applicable since we see from (2) that $a_{nN}(r)$ is continuous) that if a is less than a suitably chosen $A_2(\epsilon)$, then

$$\left| 2\pi\int_0^R P_N(r,r/a)rdr - 2\pi\int_0^R a_{0N}(r)rdr \right| < \epsilon/3.$$

Since it follows from (2) that $2\pi\int_0^R a_{0N}(r)rdr = \int_0^{2\pi}\int_0^R f(r,\phi)rdrd\phi$, we can now conclude from the above inequalities that when $a < A_1(\epsilon)$, $a < A_2(\epsilon)$,

$$\left| 2\pi\int_0^R f(r,r/a)(a^2+r^2)^{1/2}dr - \int_0^{2\pi}\int_0^R f(r,\phi)rdrd\phi \right| < \epsilon,$$

and the theorem is an immediate consequence of this inequality.

Reference

1. A. Maréchal, *Mechanical integrator for studying the distribution of light in the optical image*, Journal of the Optical Society of America vol. 37 (1947) pp. 403–404.

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